Isogeometric Analysis:
Introduction and Overview

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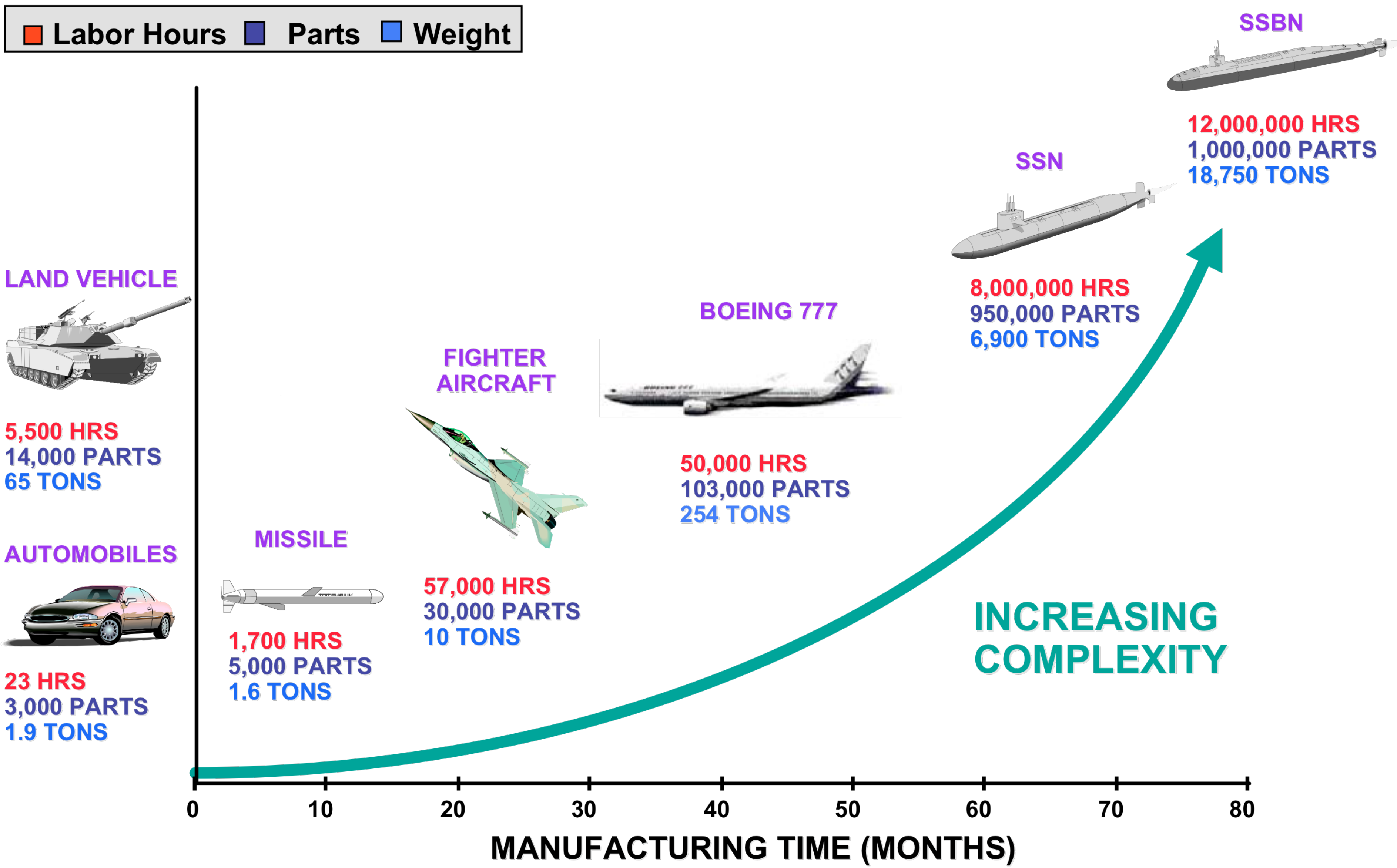
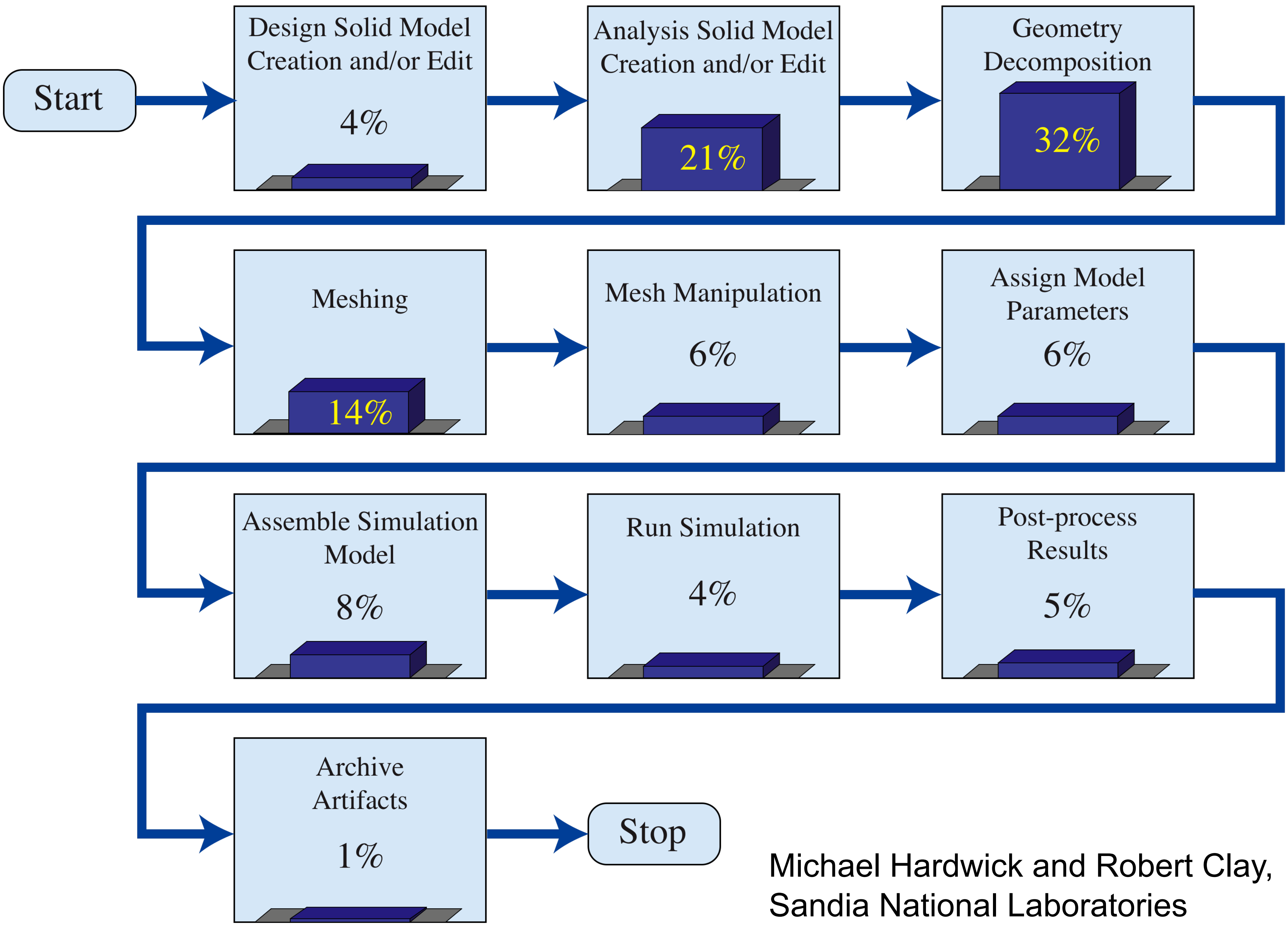
G. Sangalli, M. Scott, T. Sederberg,
C. Verhoosel, J. Zhang

LSTC
Livermore Software
Technology Corp.



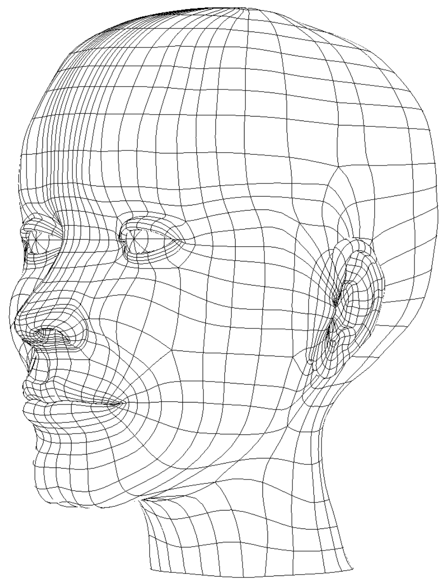
Courtesy of General Dynamics / Electric Boat Corporation



Michael Hardwick and Robert Clay,
Sandia National Laboratories

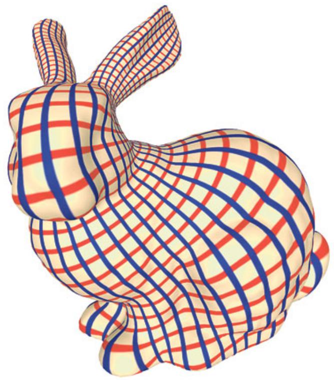
Outline

- Isogeometric analysis
- B-splines, NURBS
  - Mathematical theory of *h*-refinement
  - Structures
  - Vibrations
  - Wave propagation
  - Kolmogorov *n*-widths
  - Nonlinear solids
    - Hyperelastic nearly-incompressible solids
    - Hyperelastic-plastic solids
  - Design-through-analysis
    - Shells (w/wo rotations)
  - Fluids and fluid-structure interaction
  - Phase-field modeling
  - Cardiovascular simulation
- T-splines
  - Design-through-analysis
    - Shells
  - Nonlocal and gradient-enhanced damage-elastic materials
  - Local refinement
  - Cohesive zone analysis of discrete cracks
- Bezier extraction
- Research progress

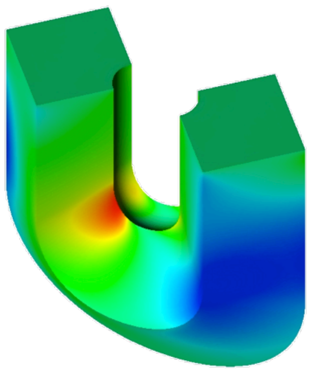


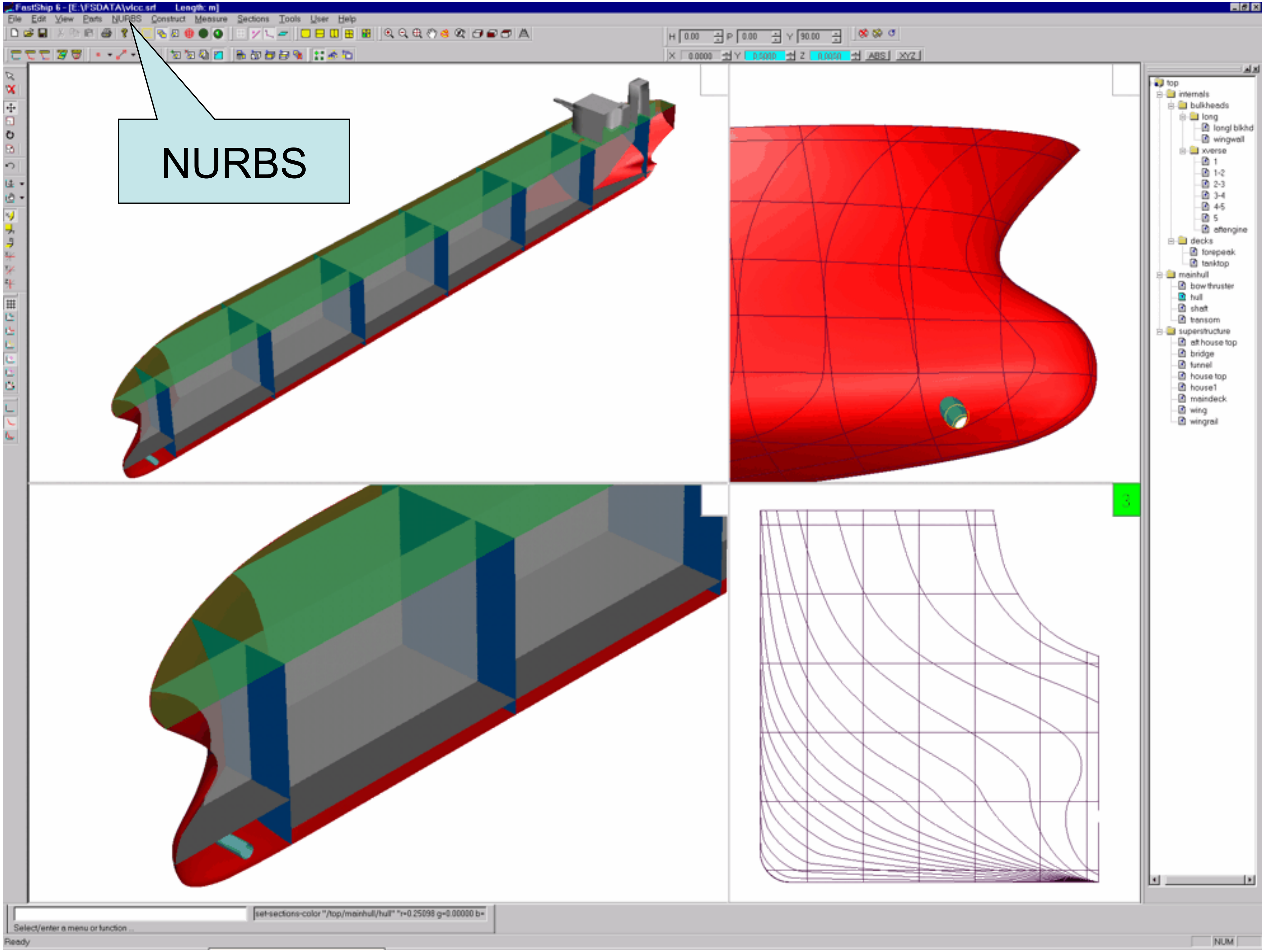
Isogeometric Analysis

- Based on technologies (e.g., NURBS, T-splines, etc.) from *computational geometry* used in:
  - Design
  - Animation
  - Graphic art
  - Visualization



- Includes standard FEA as a special case, but offers other possibilities:
  - Precise and efficient geometric modeling
  - Simplified mesh refinement
  - Smooth basis functions with compact support
  - Superior approximation properties
  - Accurate derivatives and stresses
  - *Integration* of design and analysis

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NURBS
top
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4-5
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decks
forepeak
tanktop
mainhull
bow thruster
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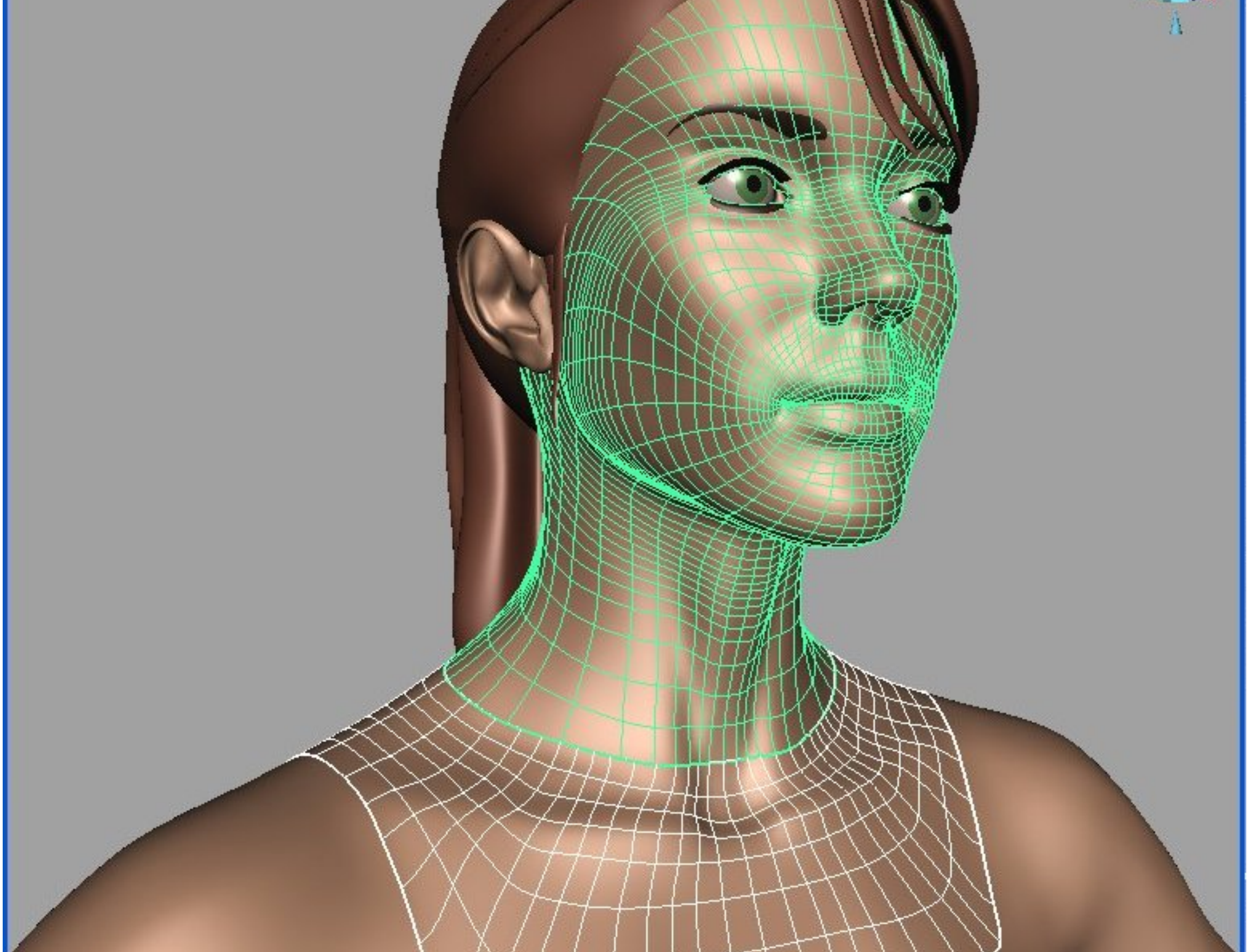
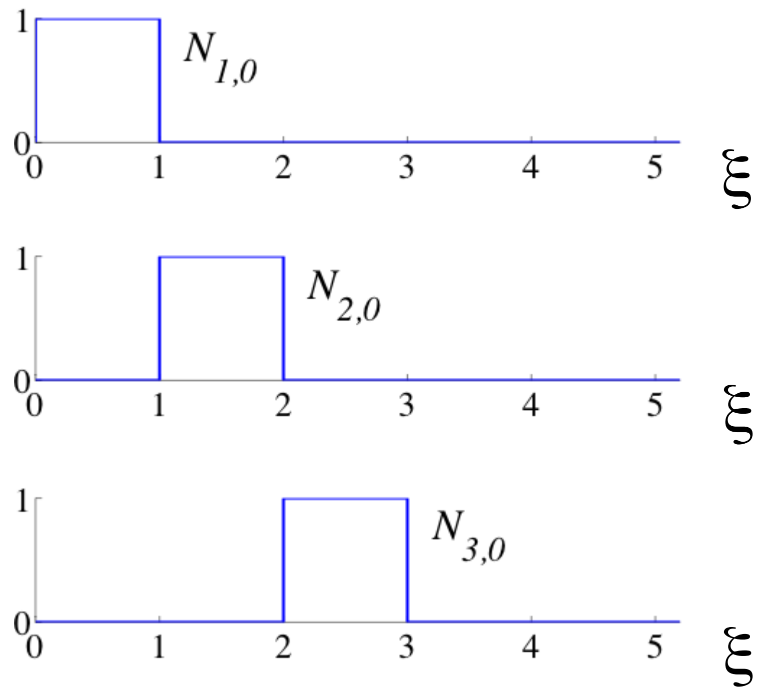
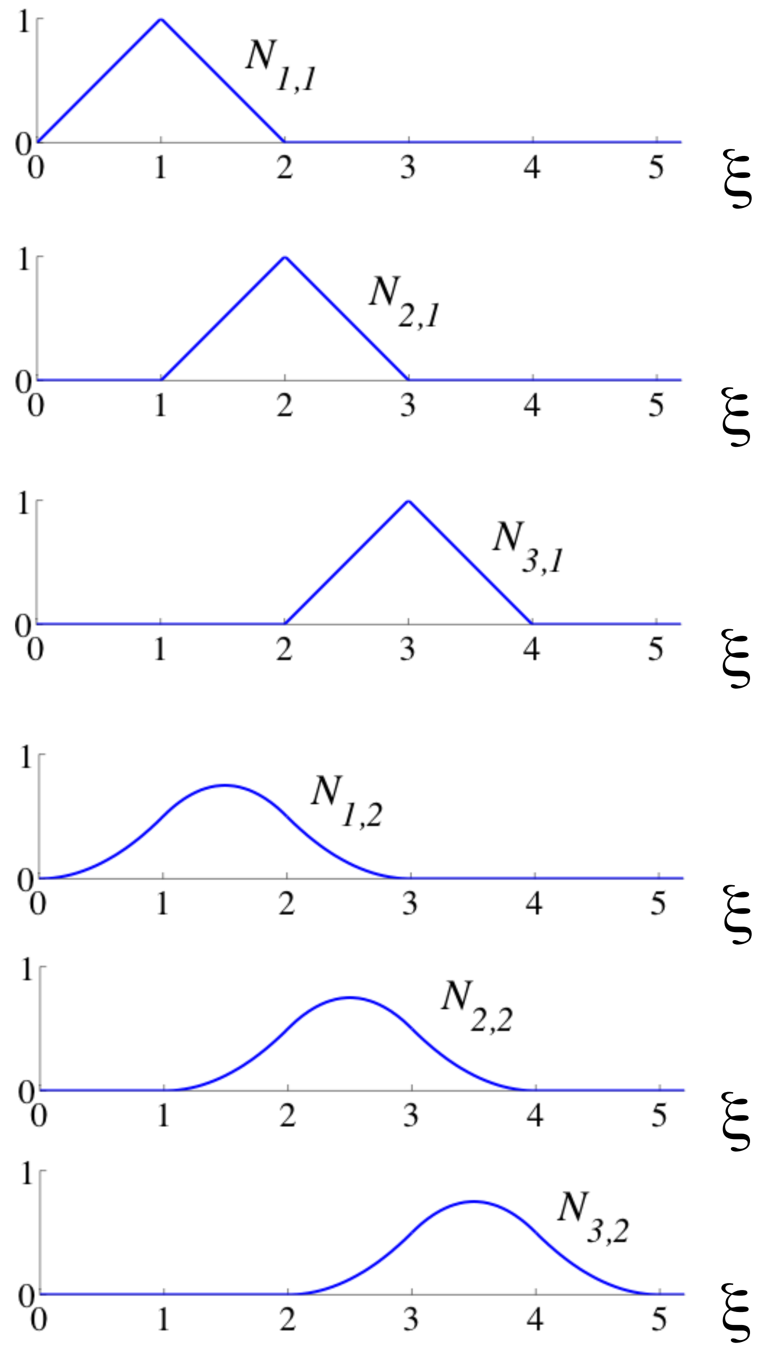


B-Splines

B-spline Basis Functions

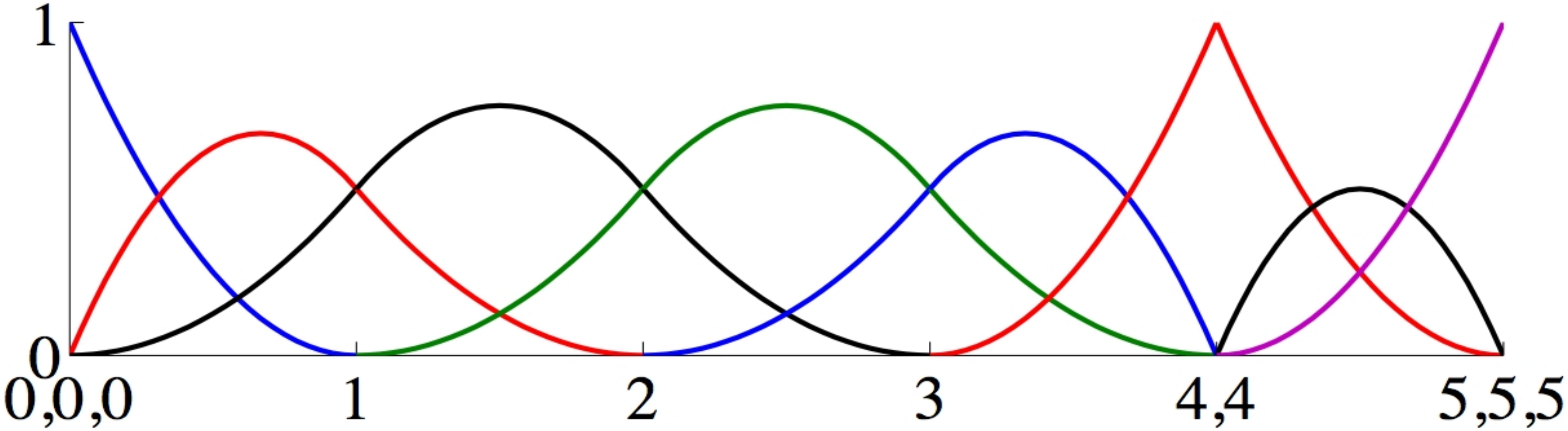
- $$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$
- $$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) \; + \; \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$





B-spline basis functions of order 0, 1, 2 for a *uniform knot vector:*

$$\Xi = \{0,1,2,3,4,\dots\}$$

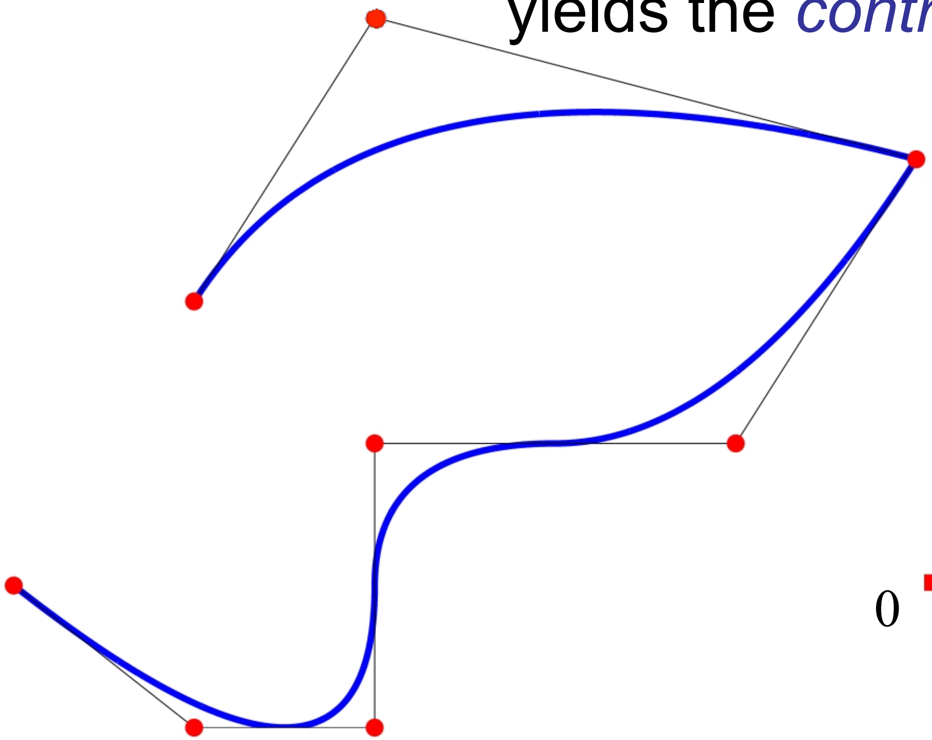


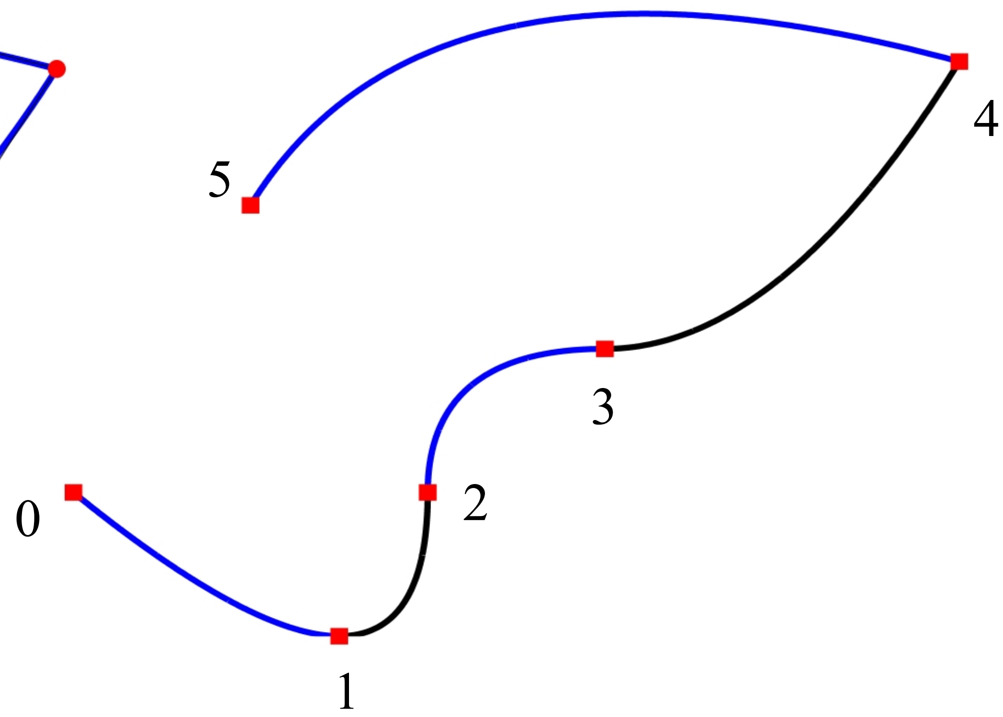
Quadratic (p=2) basis functions for an

open, non-uniform knot vector:

$$\Xi = \{0,0,0,1,2,3,4,4,5,5,5\}$$

Linear interpolation of control points yields the *control polygon*

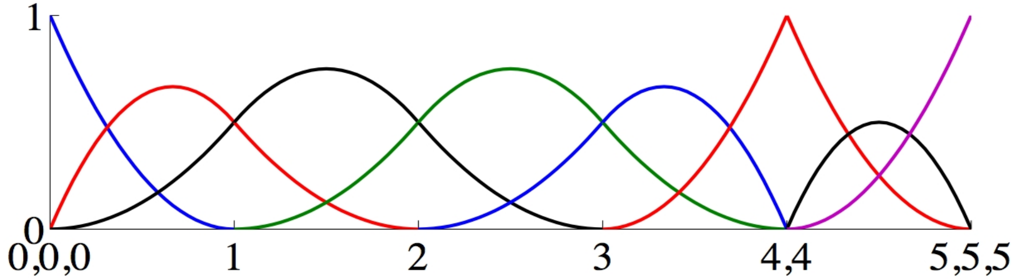




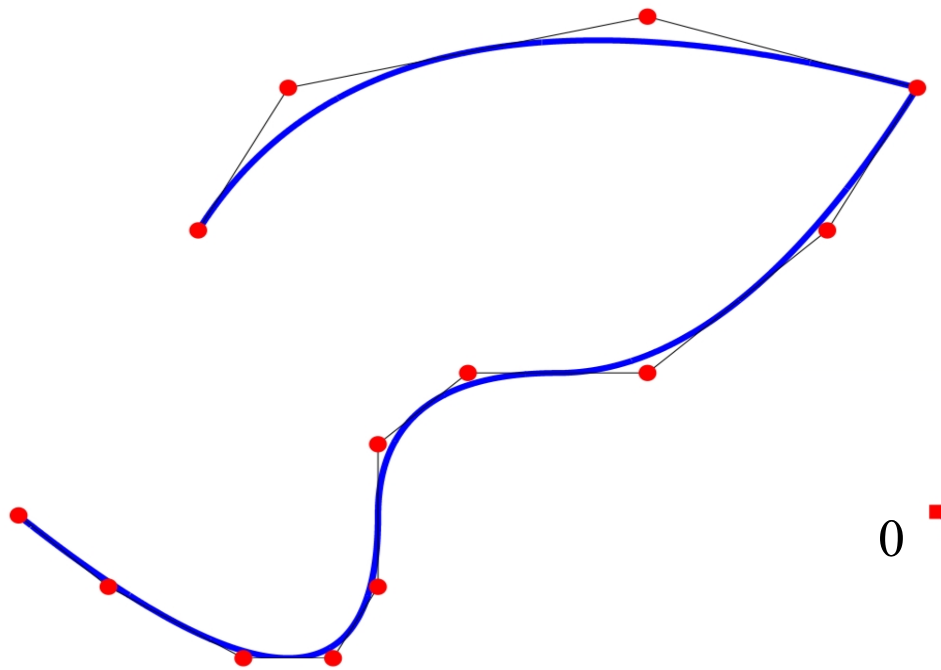
● - control points

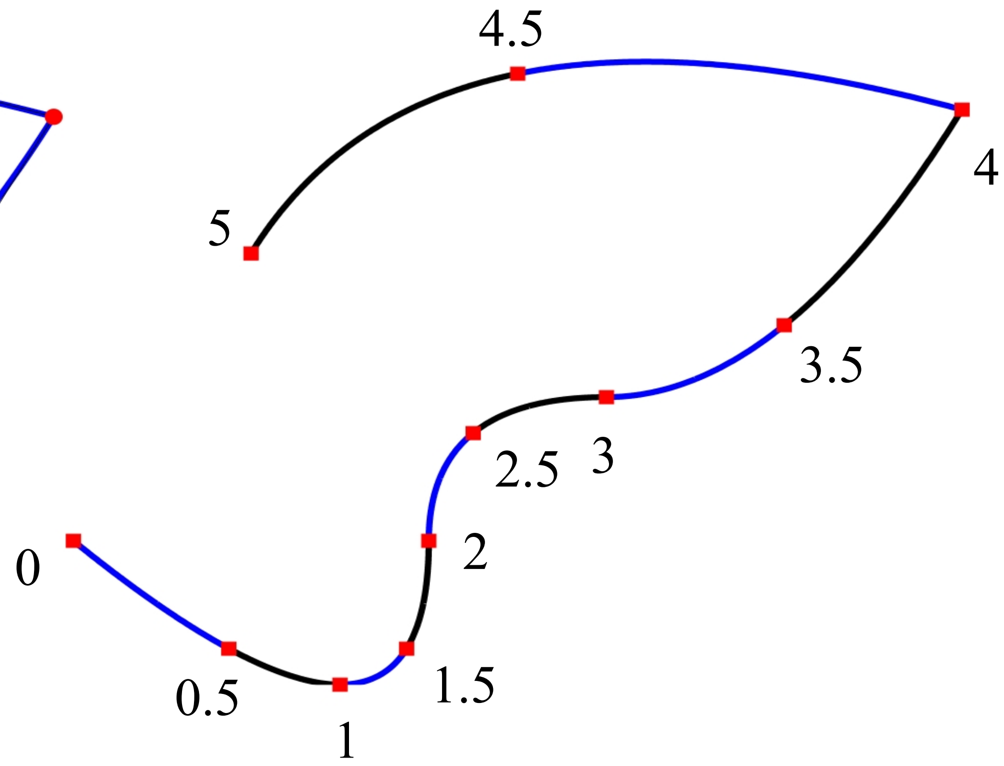
■ - knots

Quadratic basis



h-refined Curve

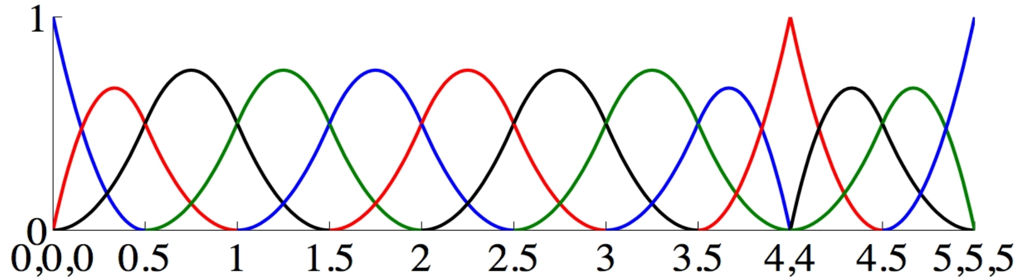




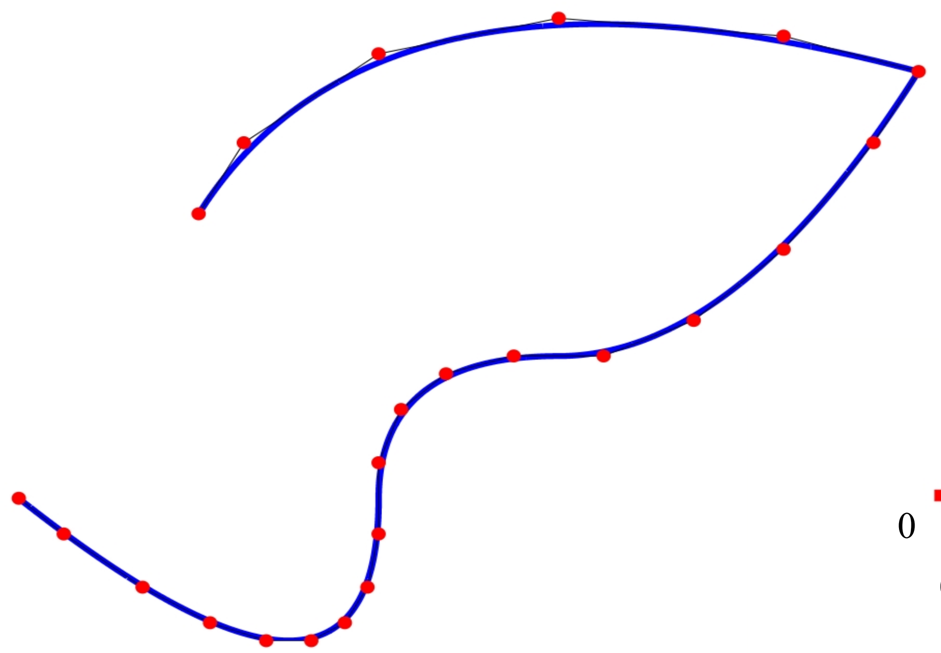
● - control points

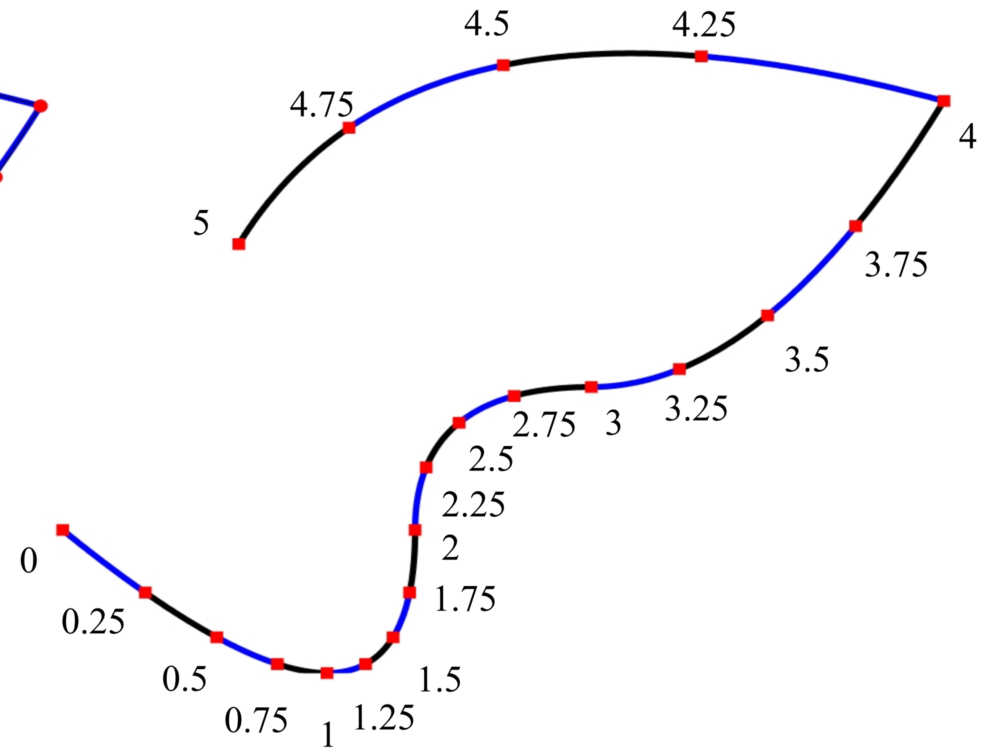
■ - knots

Quadratic basis



Further *h*-refined Curve

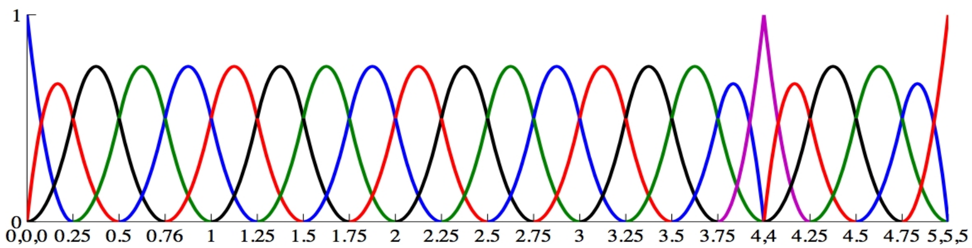




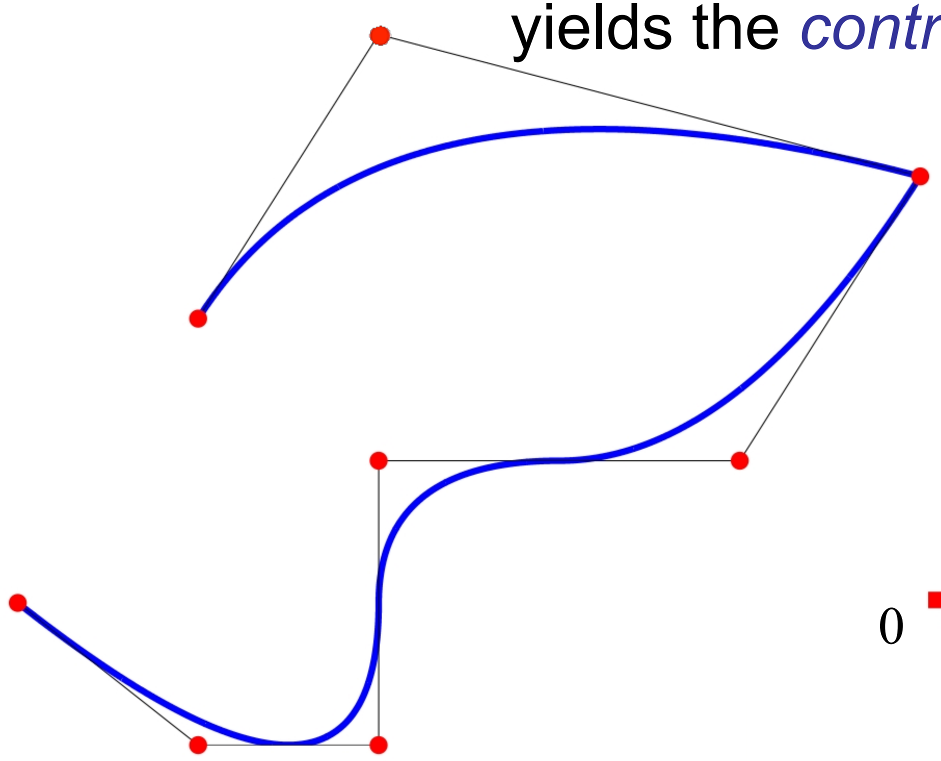
● - control points

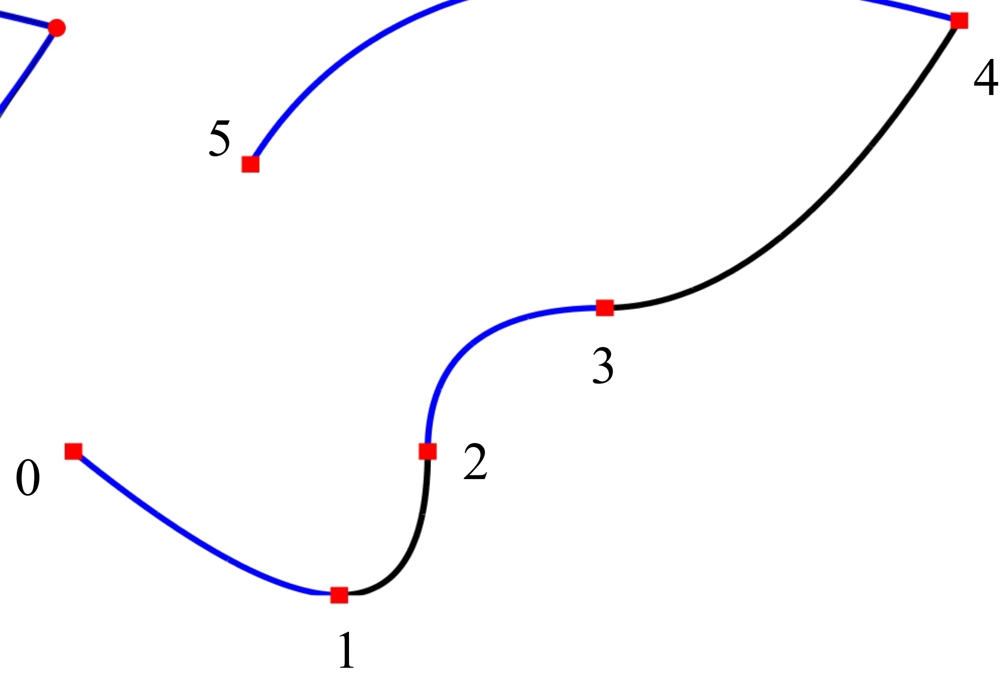
■ - knots

Quadratic basis



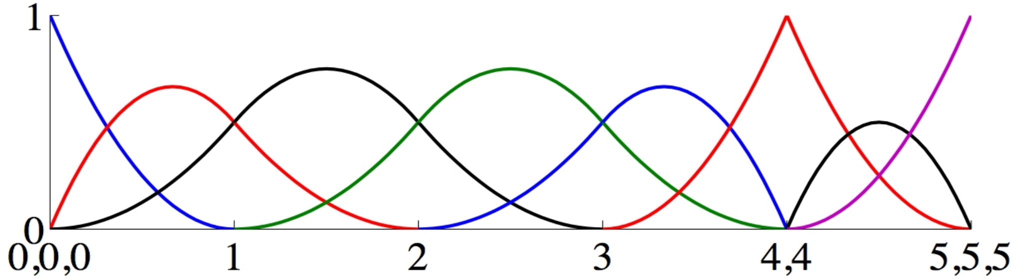
Linear interpolation of control points yields the *control polygon*



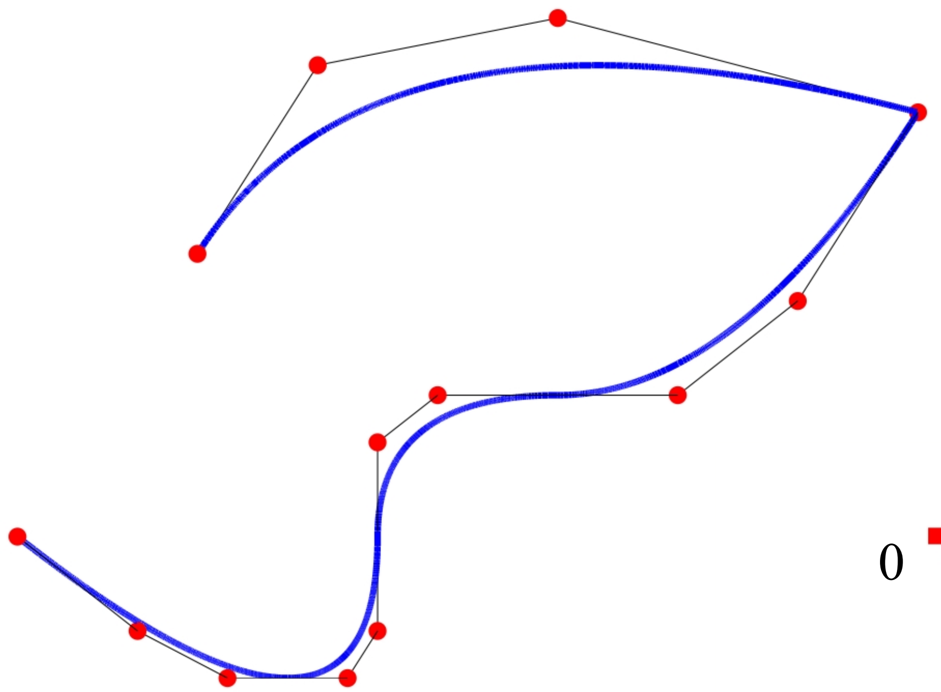


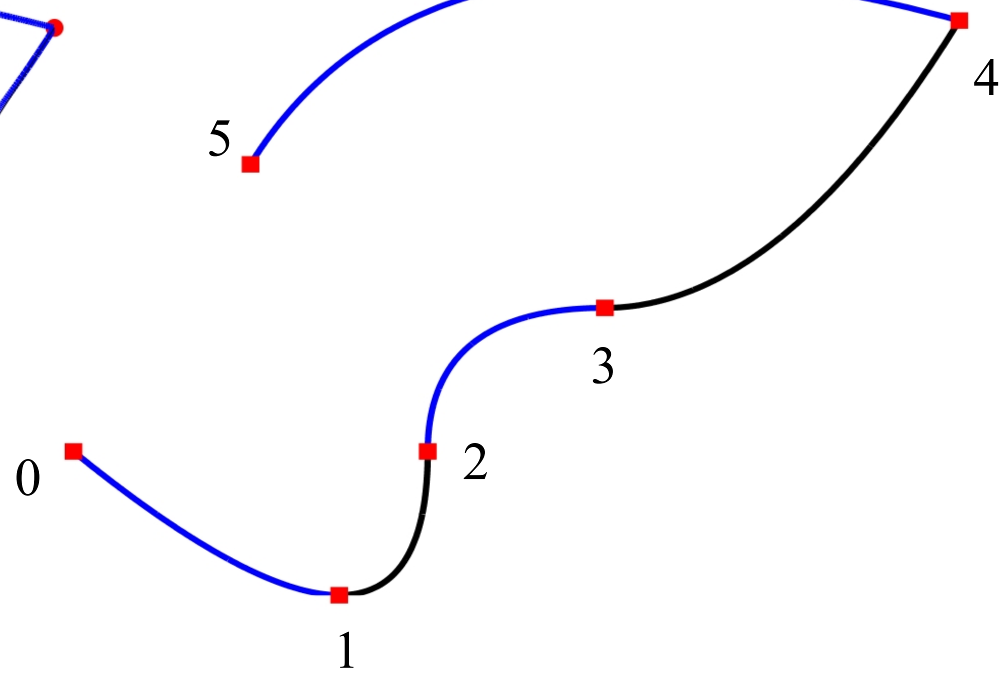
● - control points ■ - knots

Quadratic basis



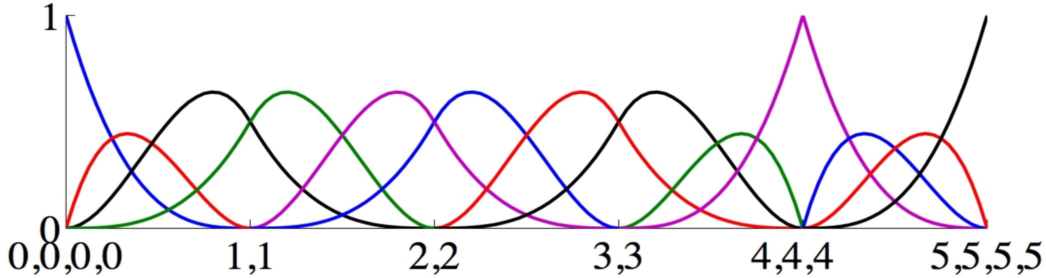
Cubic *p*-refined Curve



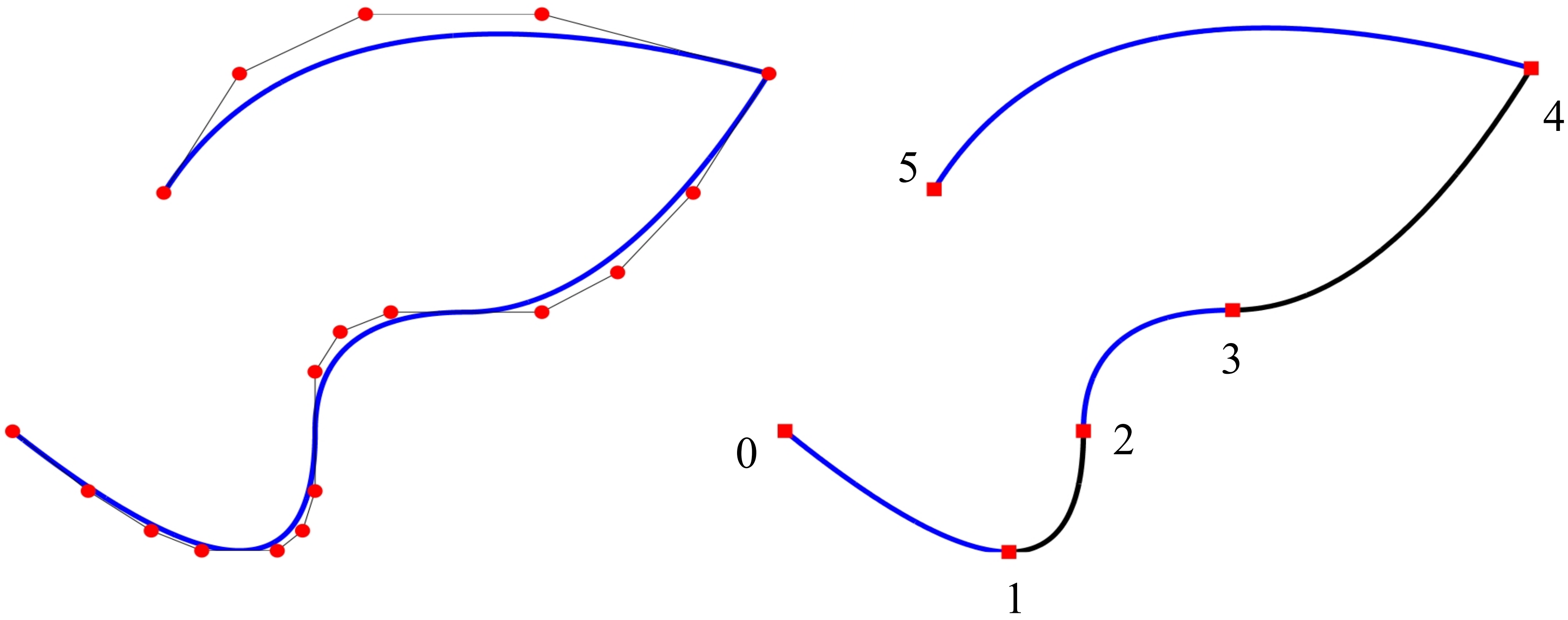


● - control points ■ - knots

Cubic basis

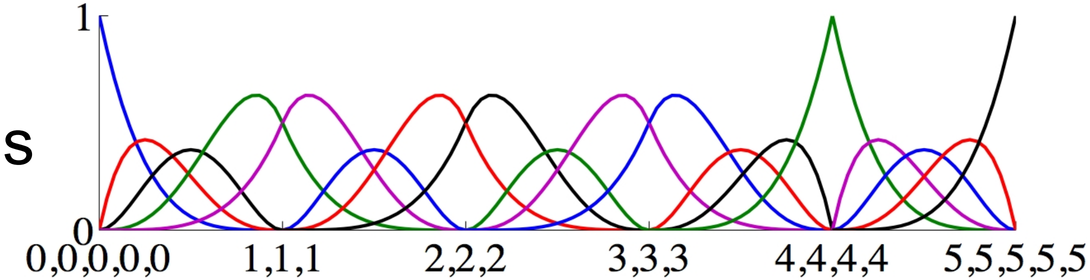


Quartic *p*-refined Curve



● - control points ■ - knots

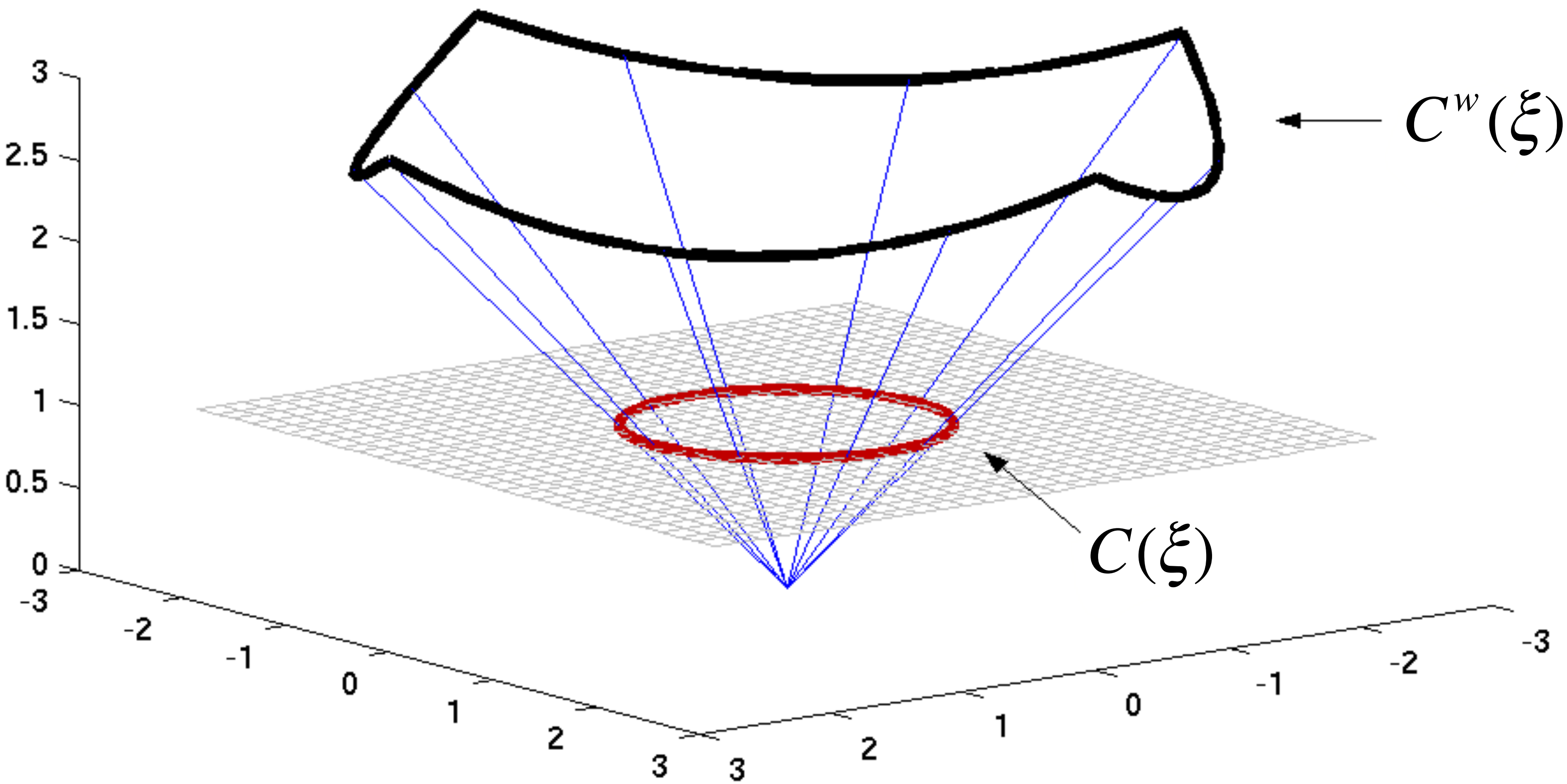
Quartic basis



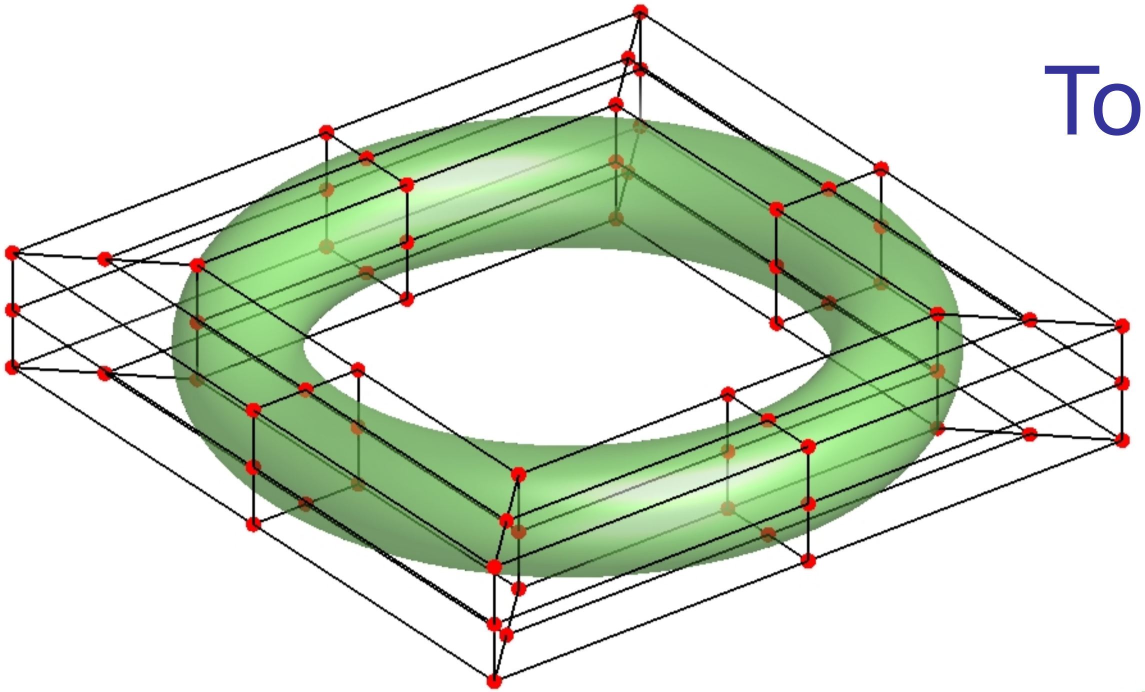
NURBS

Non-Uniform Rational B-splines

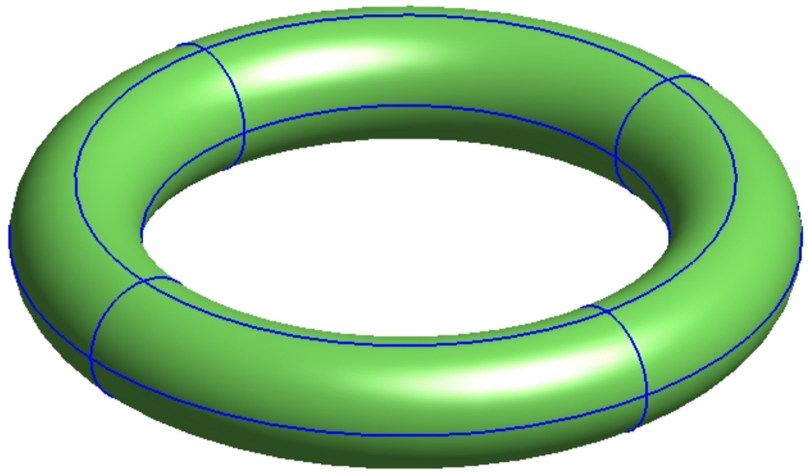
Circle from 3D Piecewise Quadratic Curves



Toroidal Surface

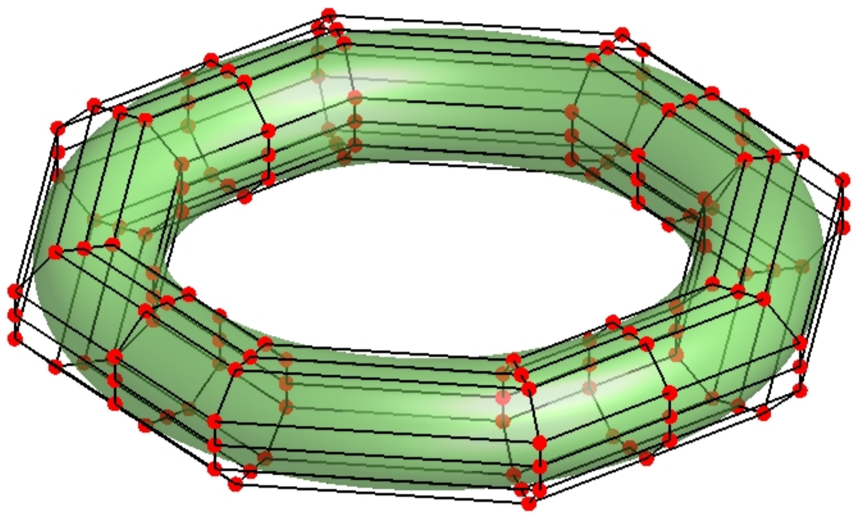


Control net

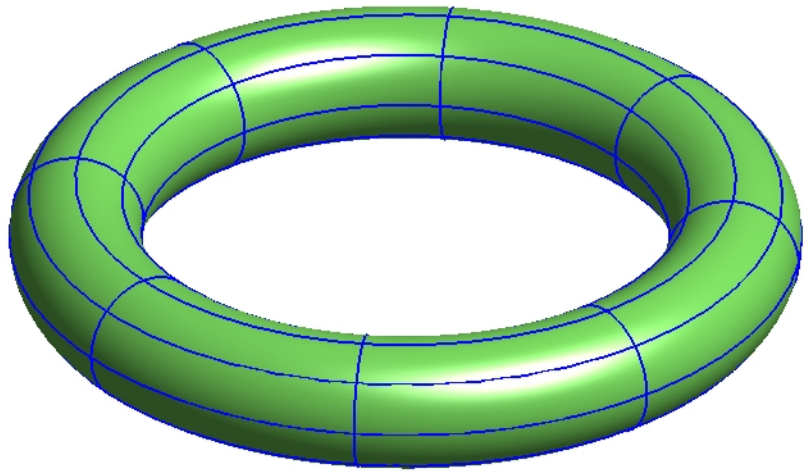


Mesh

h-refined Surface

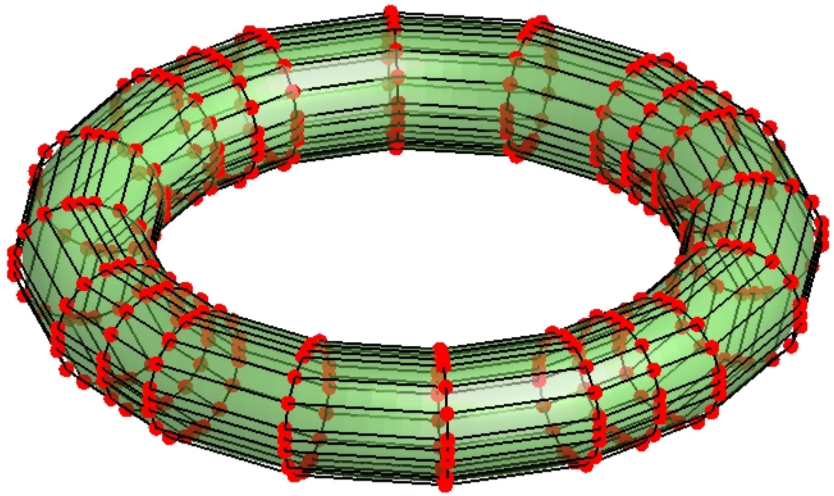


Control net

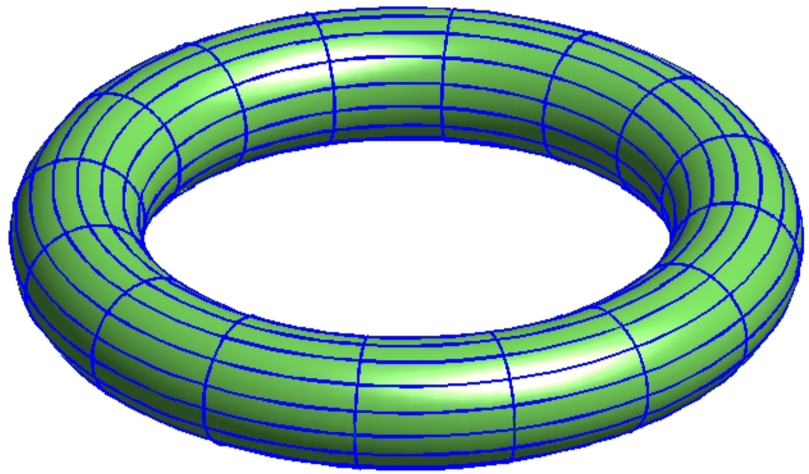


Mesh

Further *h*-refined Surface

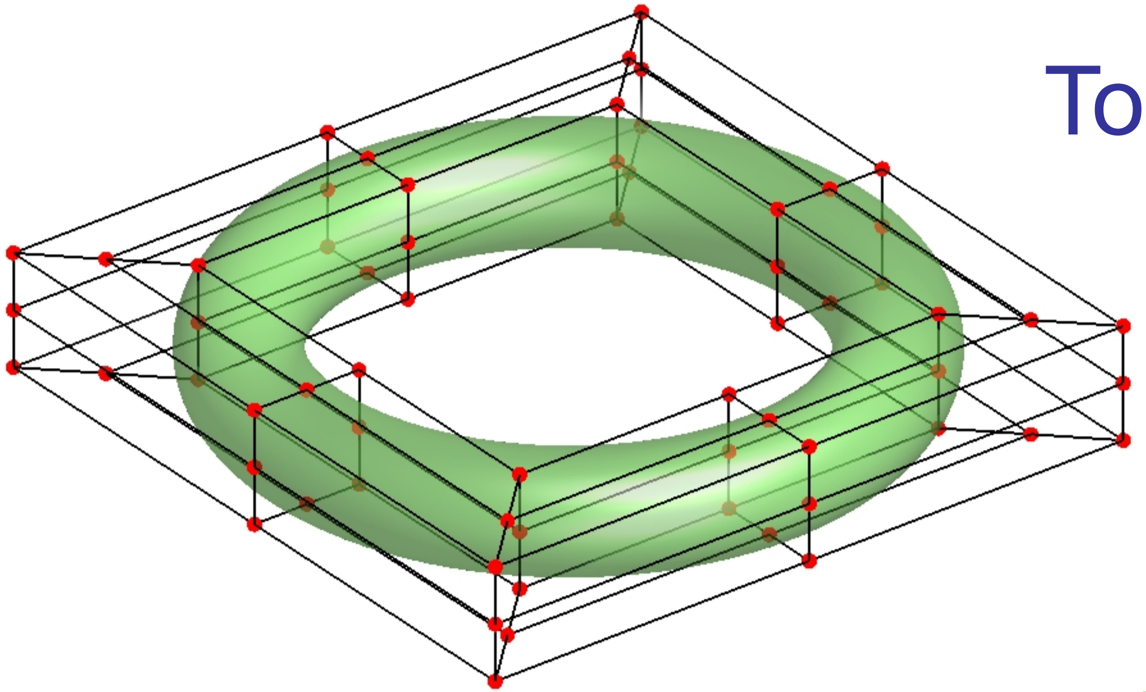


Control net

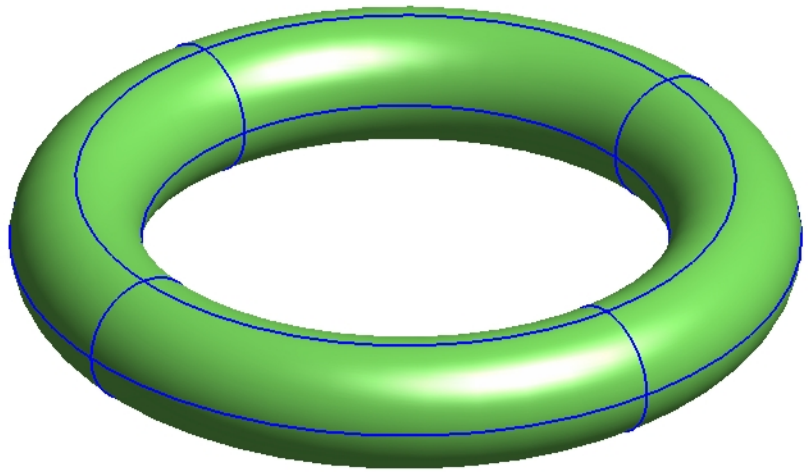


Mesh

Toroidal Surface

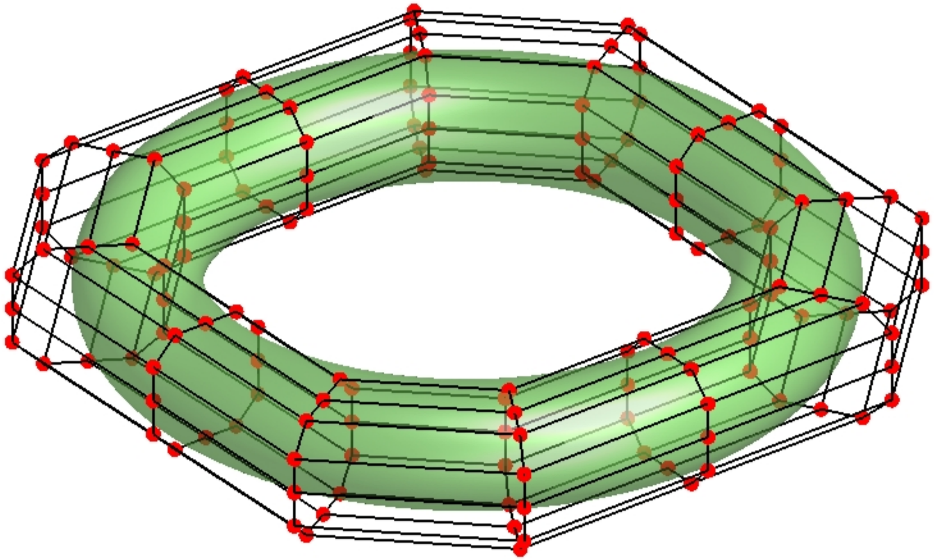


Control net

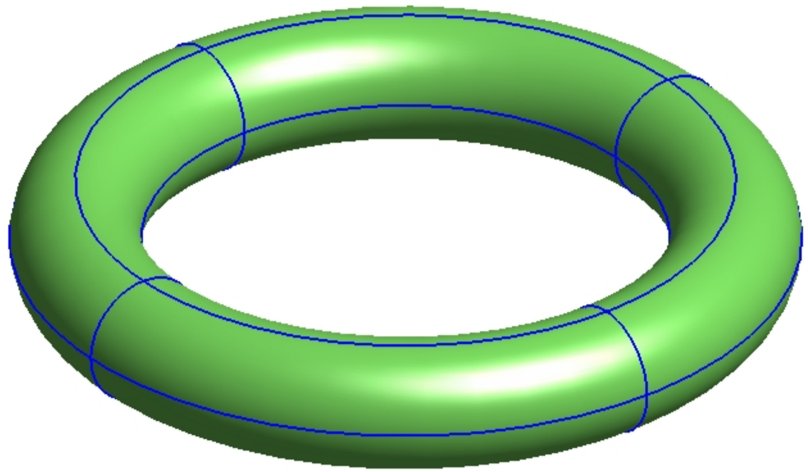


Mesh

Cubic *p*-refined Surface

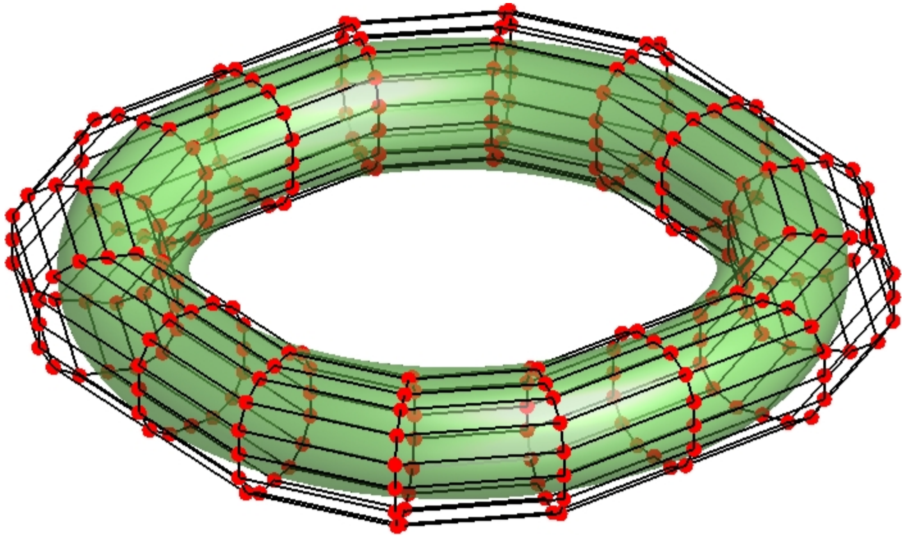


Control net

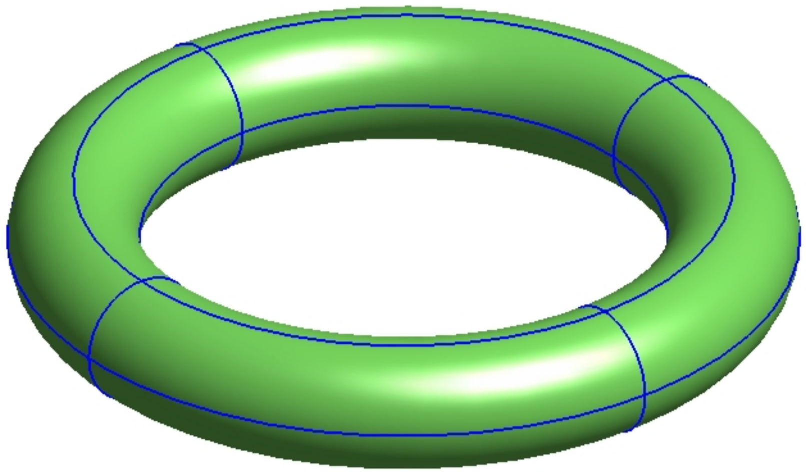


Mesh

Quartic *p*-refined Surface

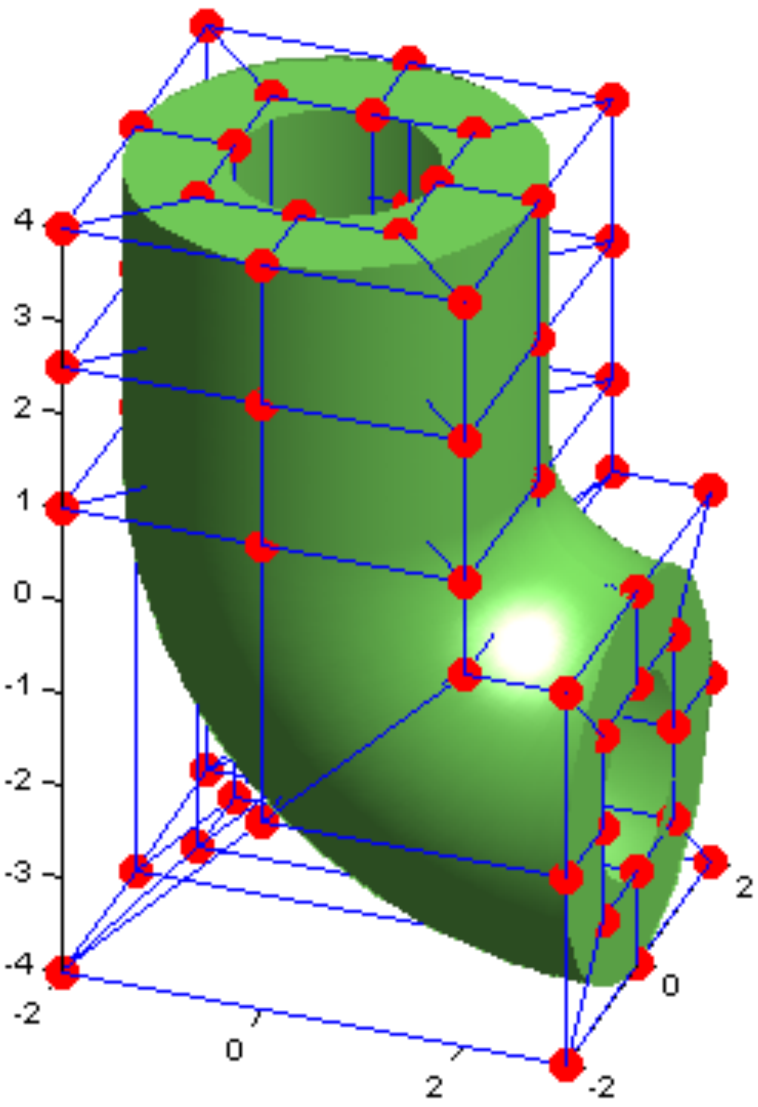


Control net

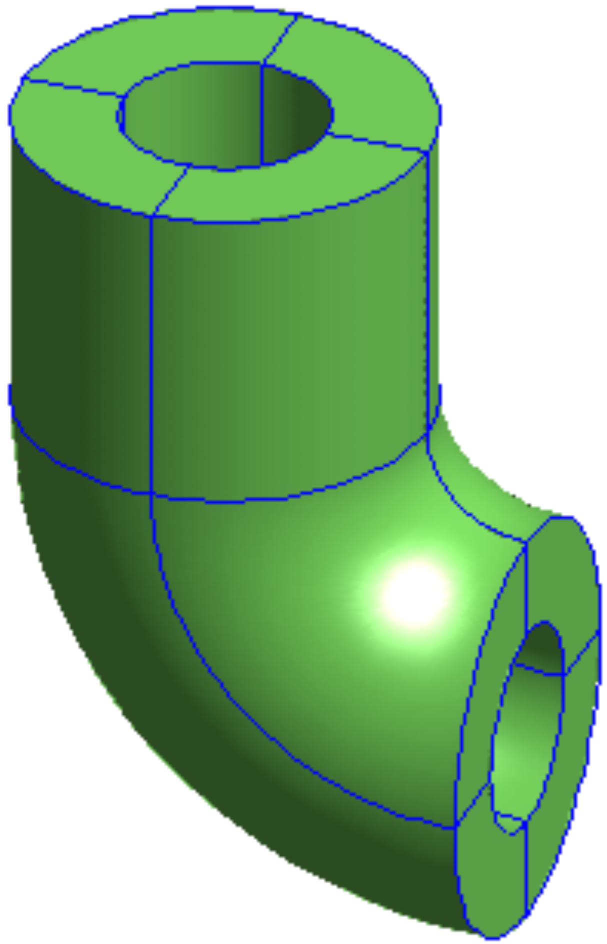


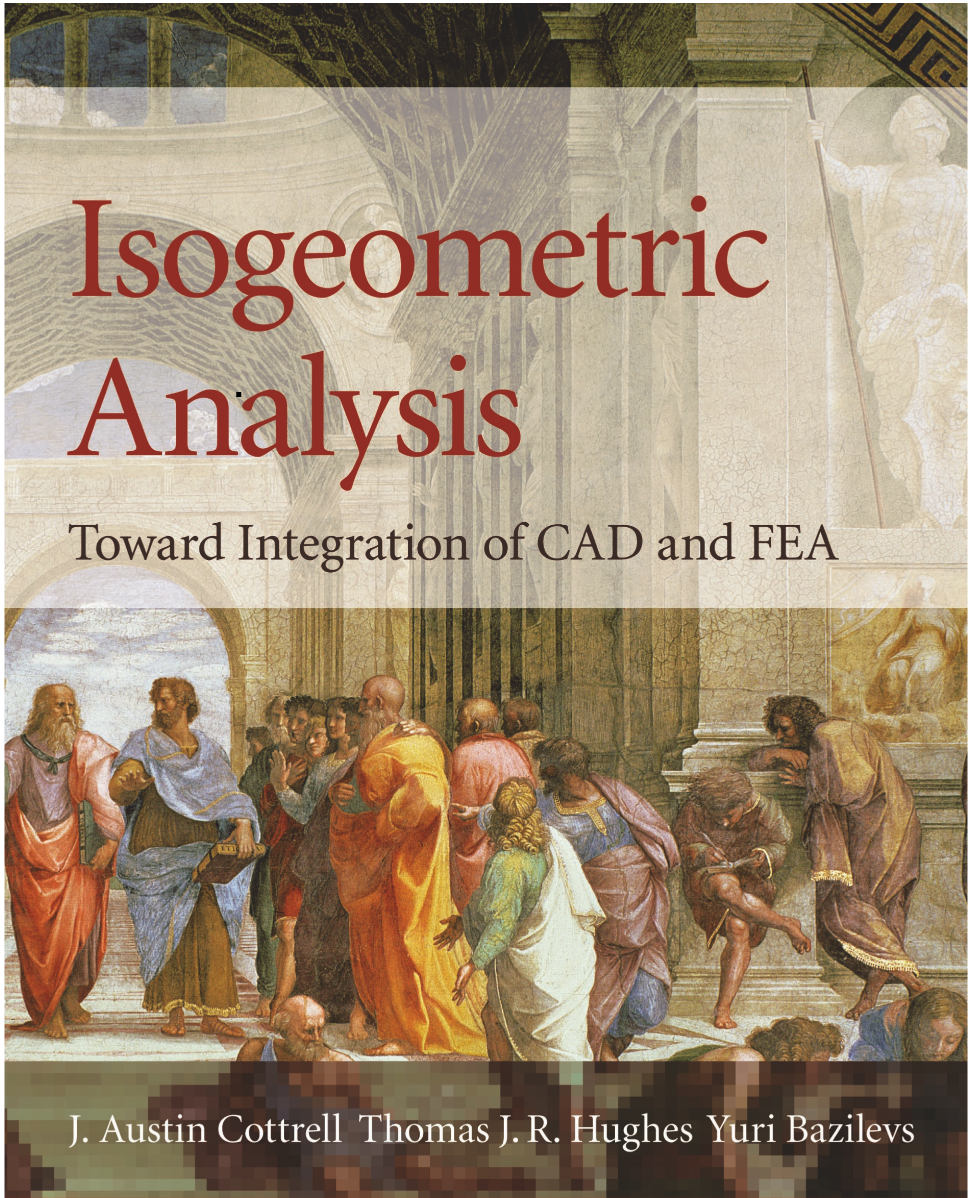
Mesh

Control Net



Mesh

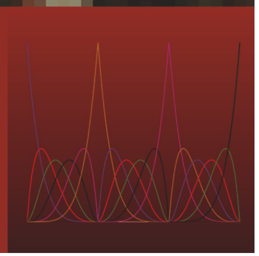




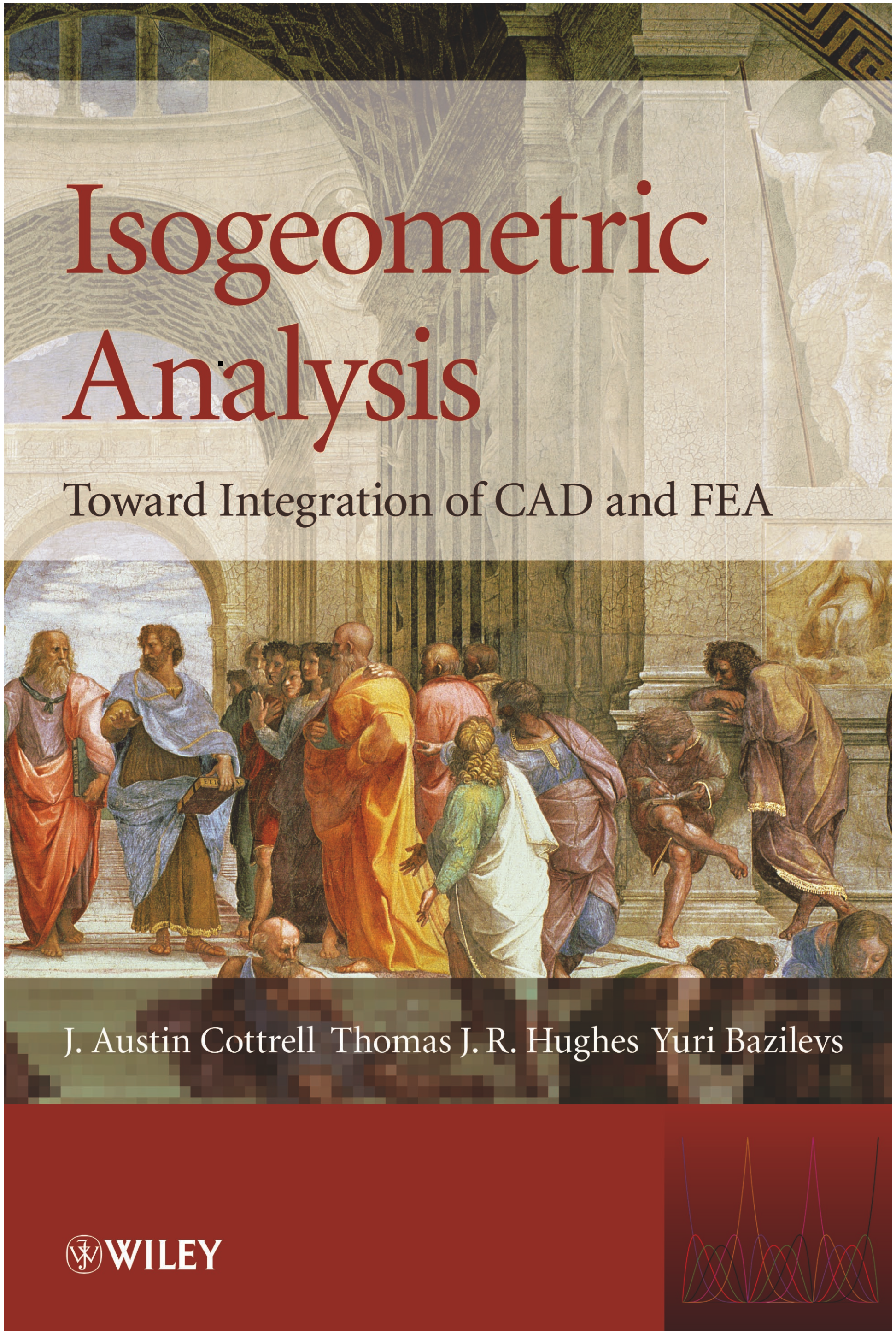
Isogeometric Analysis

Toward Integration of CAD and FEA

J. Austin Cottrell Thomas J. R. Hughes Yuri Bazilevs



WILEY



ICES

Institute for Computational
Engineering and Sciences

Austin, Texas, U.S.A.

Finite Element Analysis and Isogeometric Analysis

▪ Compact support
▪ Partition of unity
▪ Affine covariance
▪ Isoparametric concept
▪ Patch tests satisfied

Approximation with NURBS

Theorem

Let $k, l,$ be the integer indices such that $0 \le k \le l \le p+1$. Let $u \in H^l(\Omega)$, then

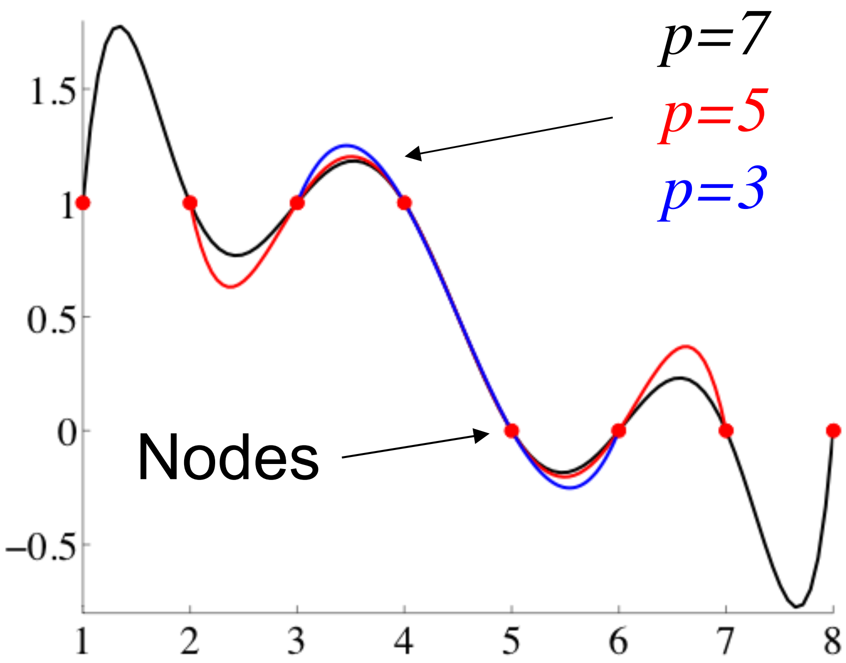
$$\sum_{K \in \mathrm{K}_h} \left| u - \Pi_{V_h} u \right|^2_{H^k(K)} \le C \sum_{K \in \mathrm{K}_h} h_K^{2(l-k)} \sum_{i=0}^{l} \left\| \nabla \mathbf{F} \right\|^{2(i-l)}_{L^\infty\left(\mathbf{F}^{-1}(K)\right)} \left| u \right|^2_{H^i(K)}$$

Positive "constant," depends on *p*, smoothness of V_h, shape regularity of the mesh, shape of Ω (but not its size), etc.

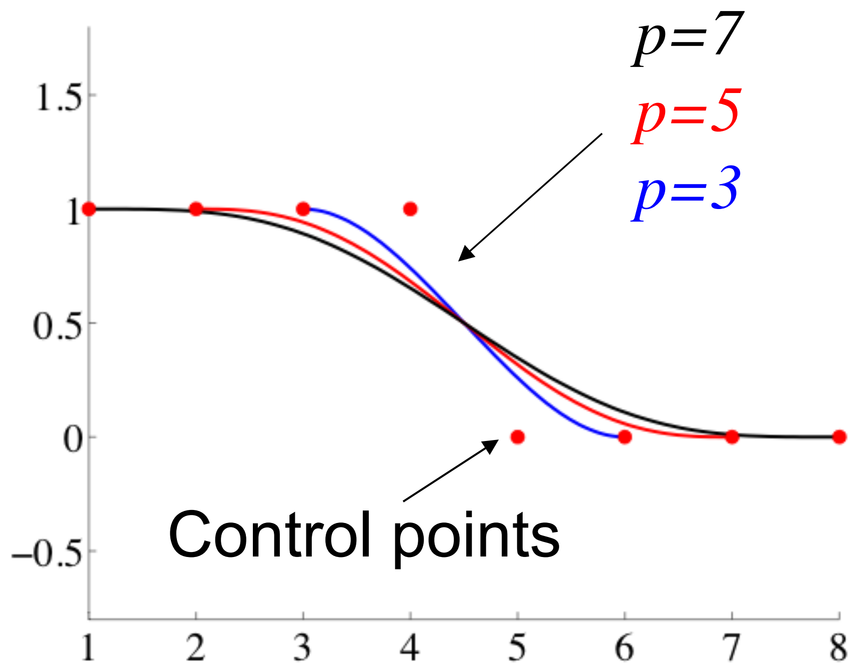
Factors which render error estimate dimensionally consistent.

Variation Diminishing Property

Lagrange polynomials

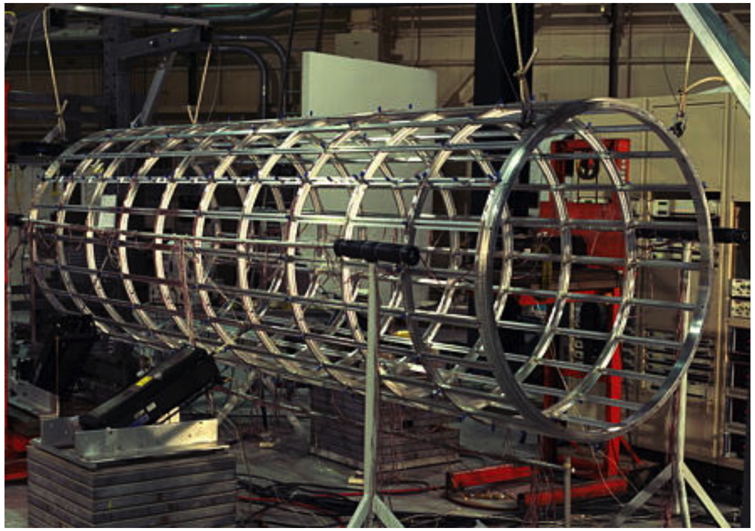


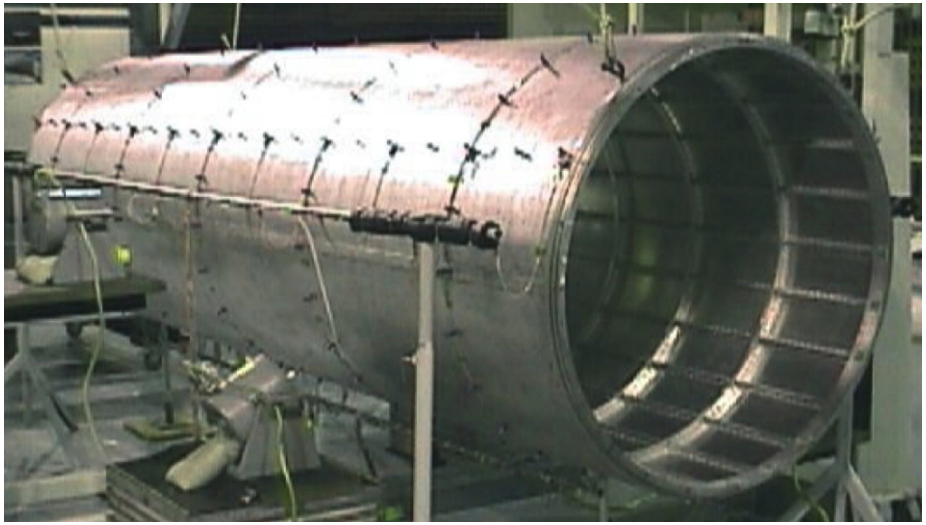
NURBS



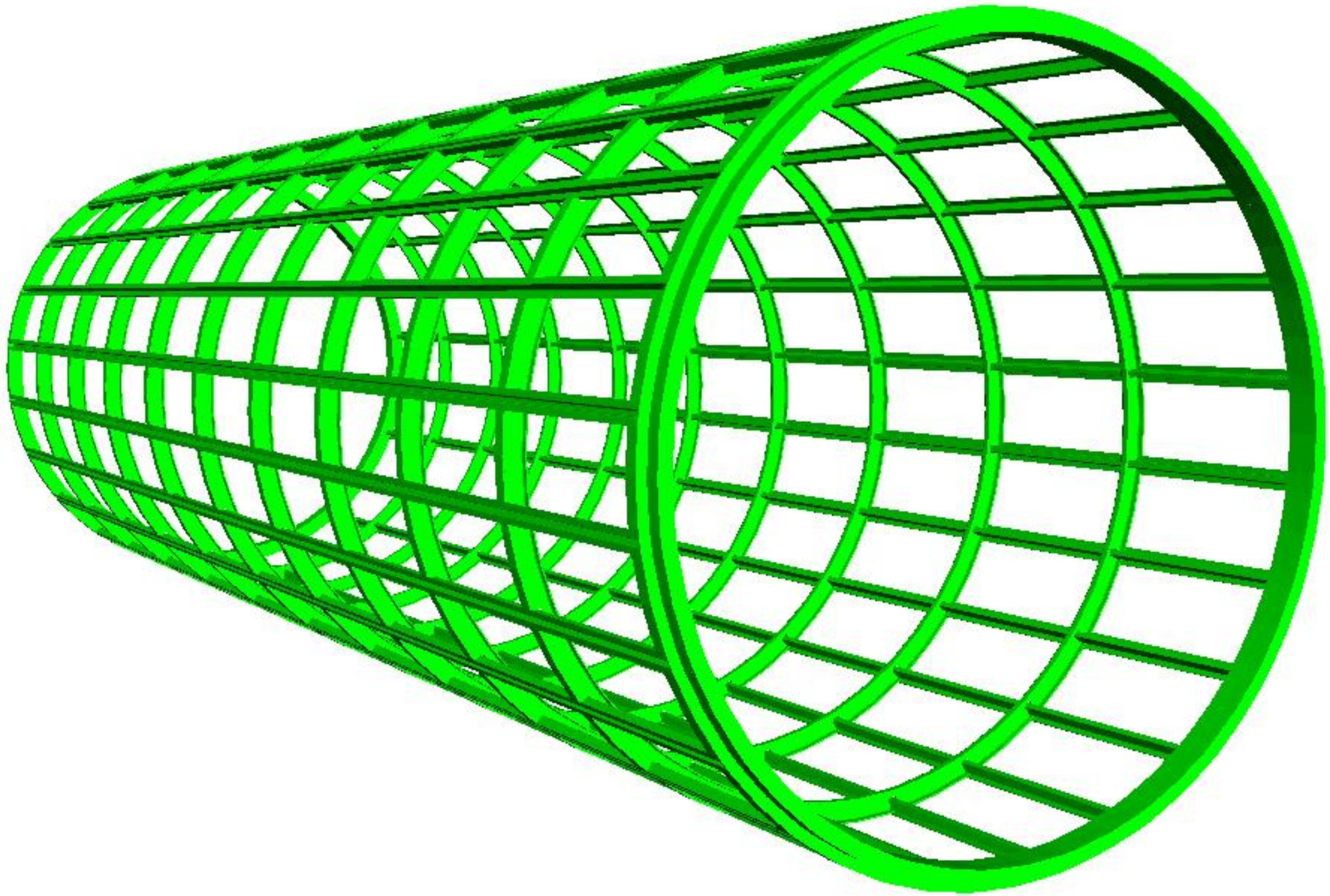
Vibration Analysis

NASA Aluminum Testbed Cylinder (ATC)

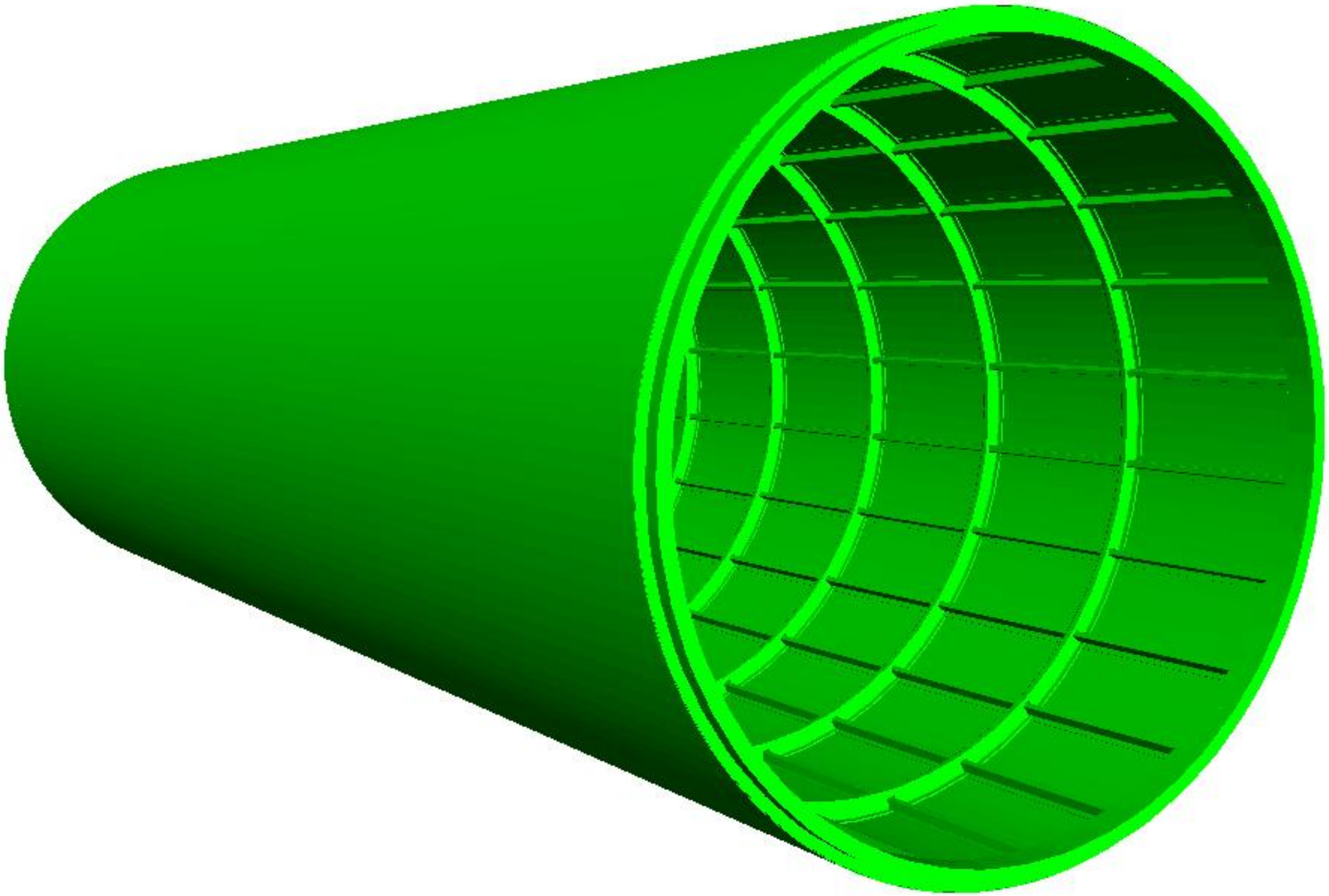




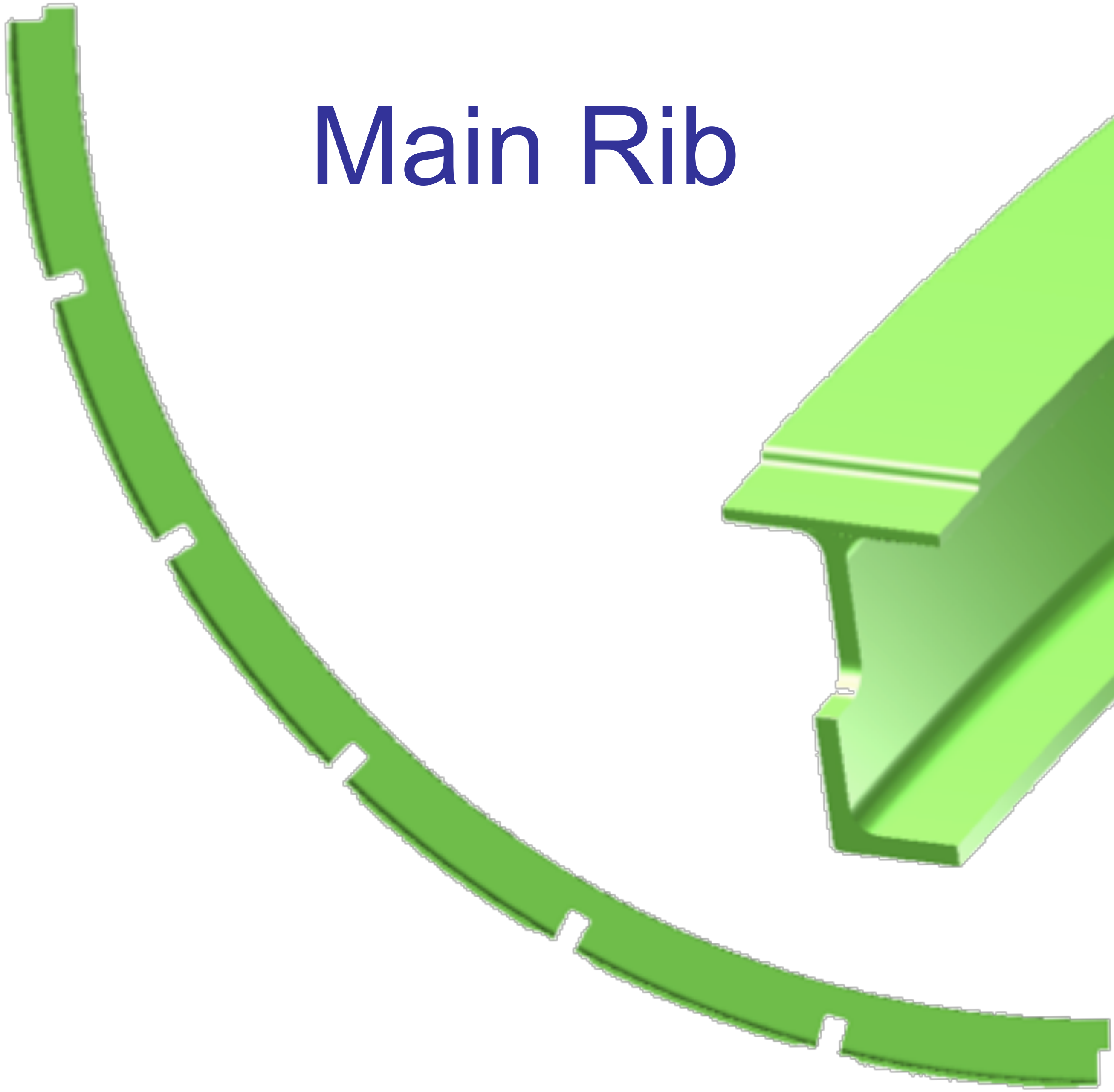
NASA ATC Frame

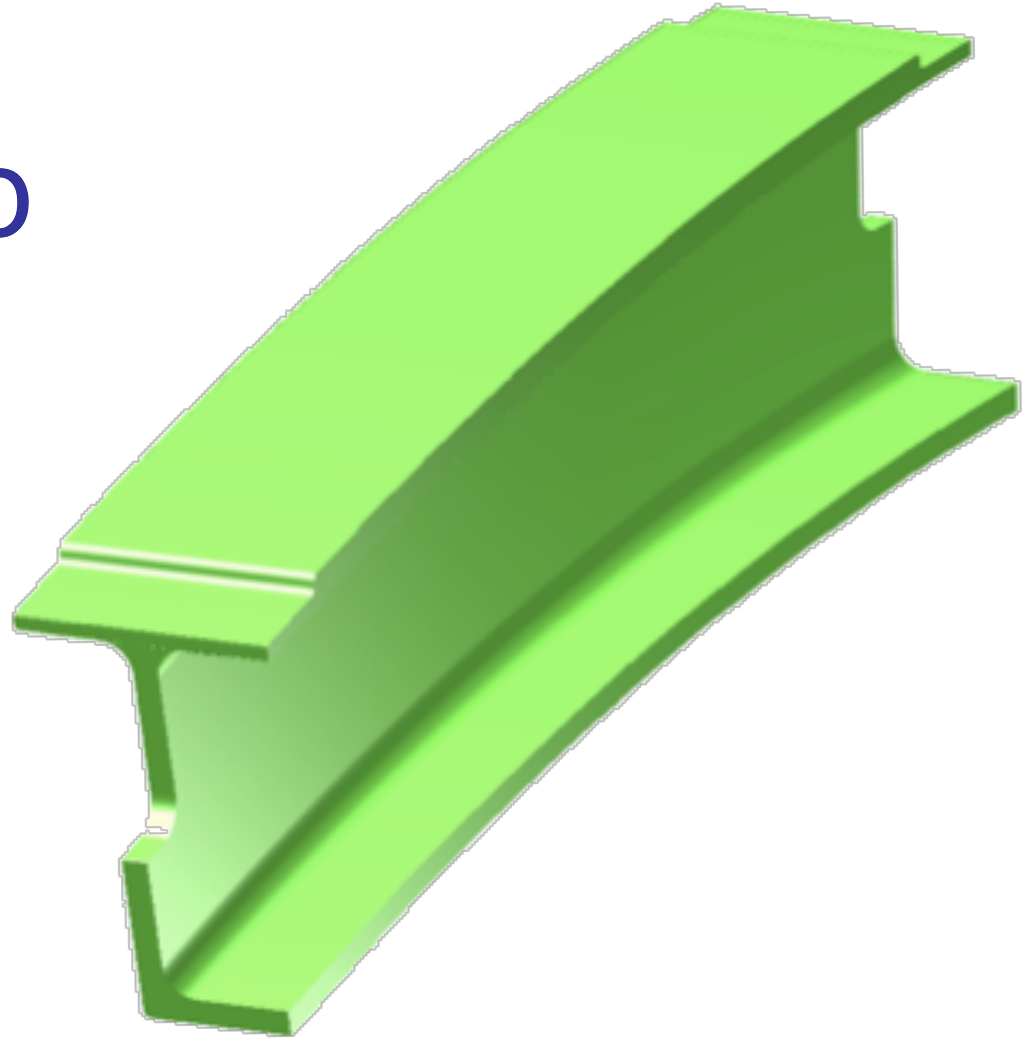


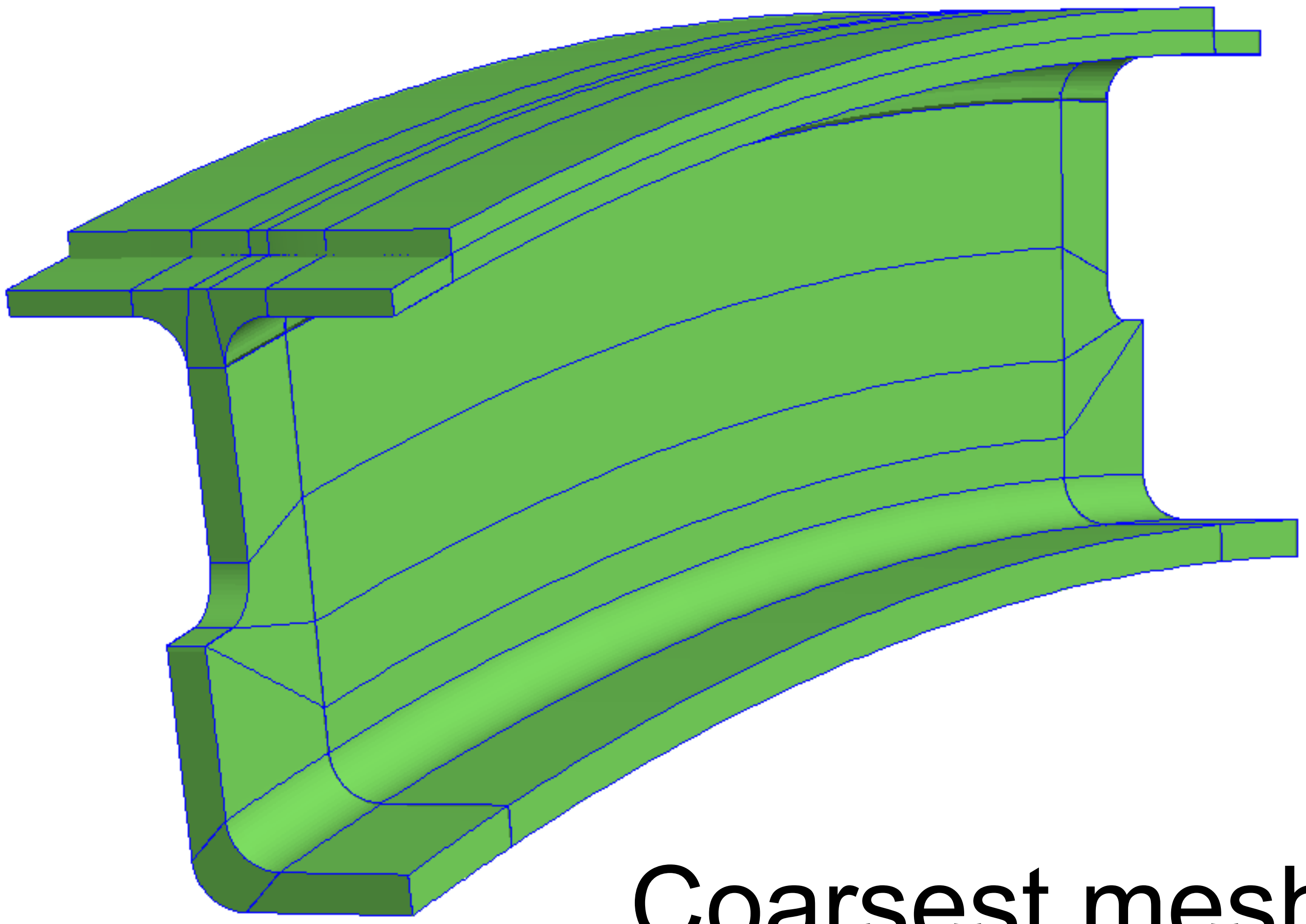
NASA ATC Frame and Skin



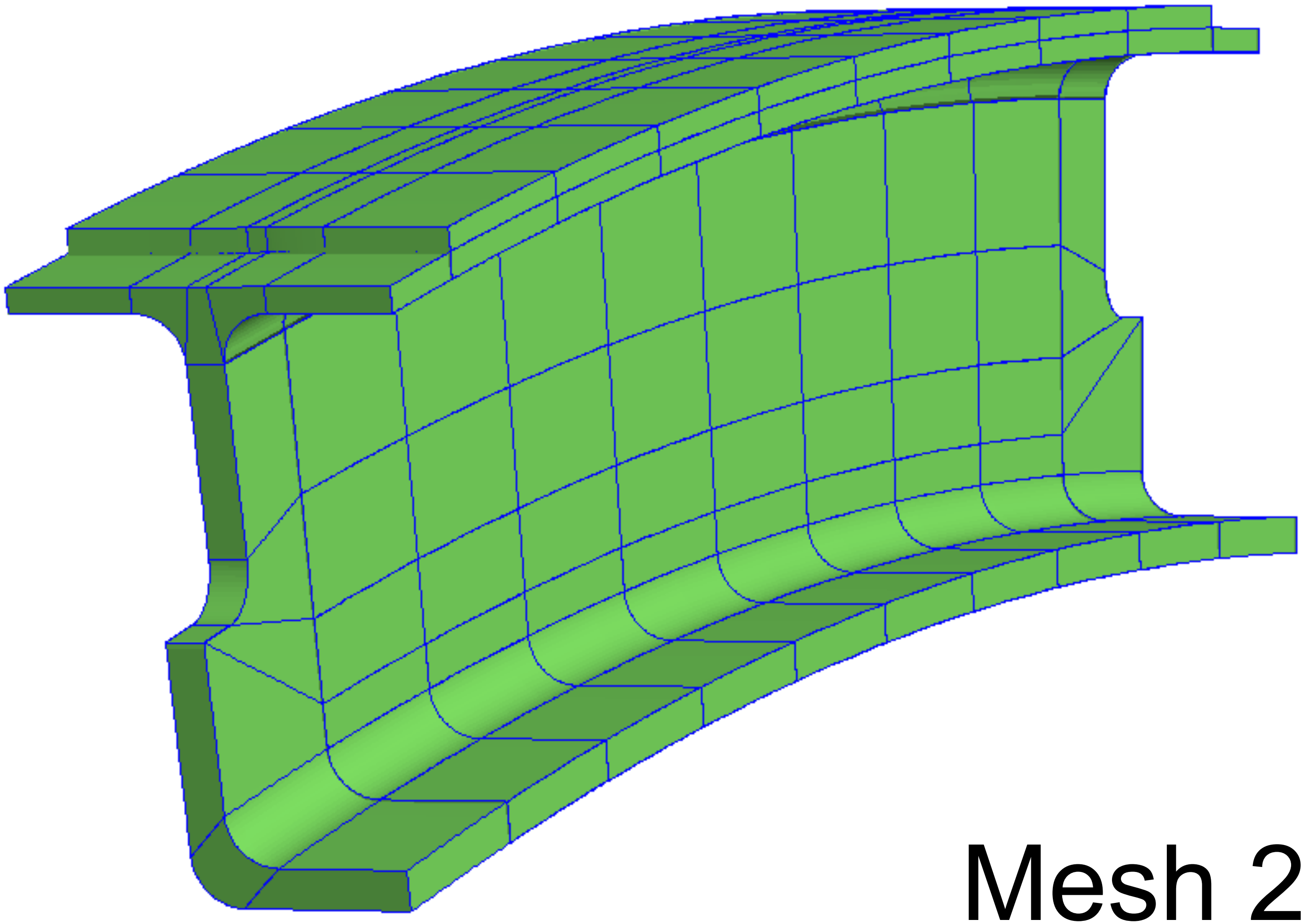
Main Rib



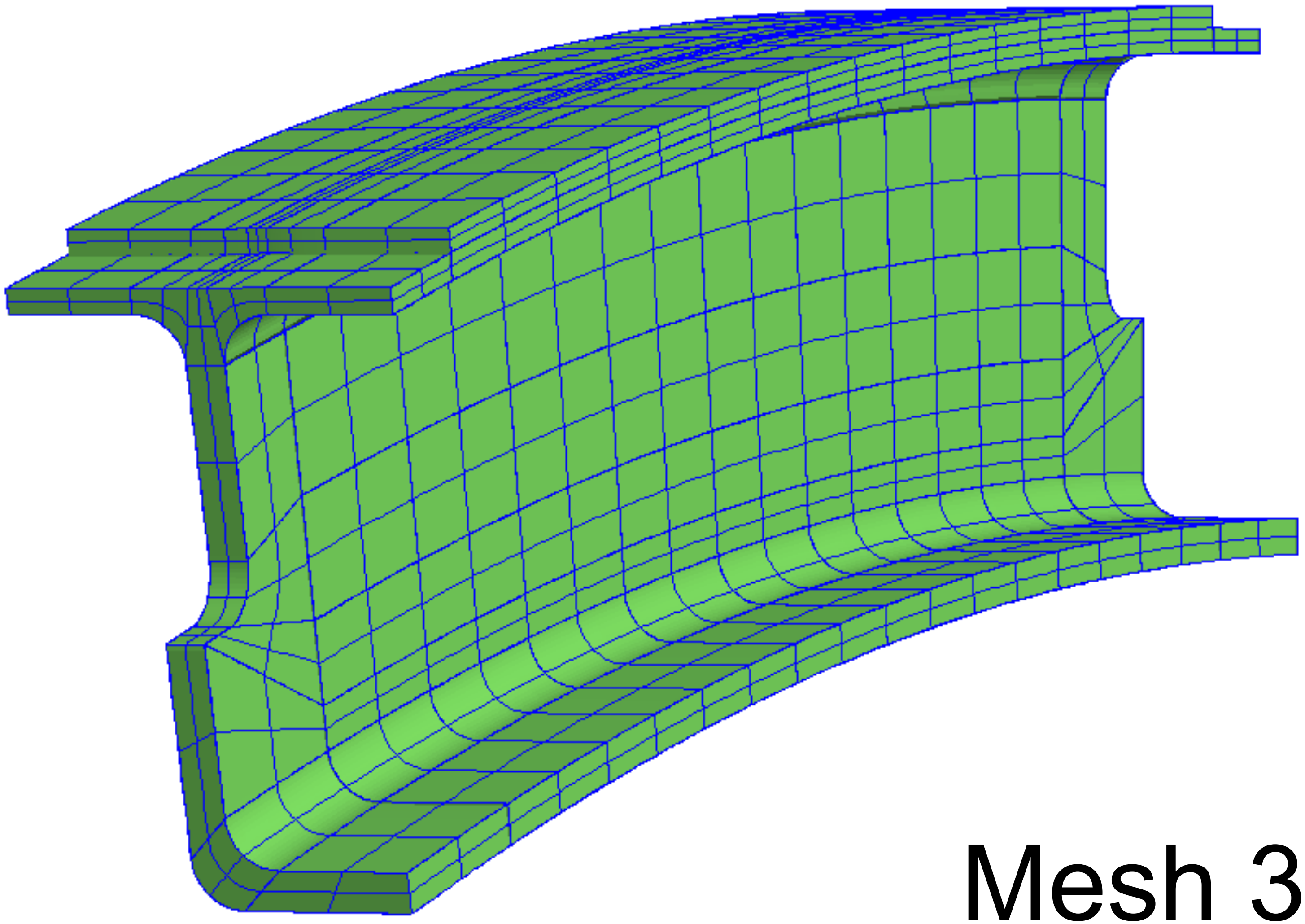




Coarsest mesh
15° segment of main rib

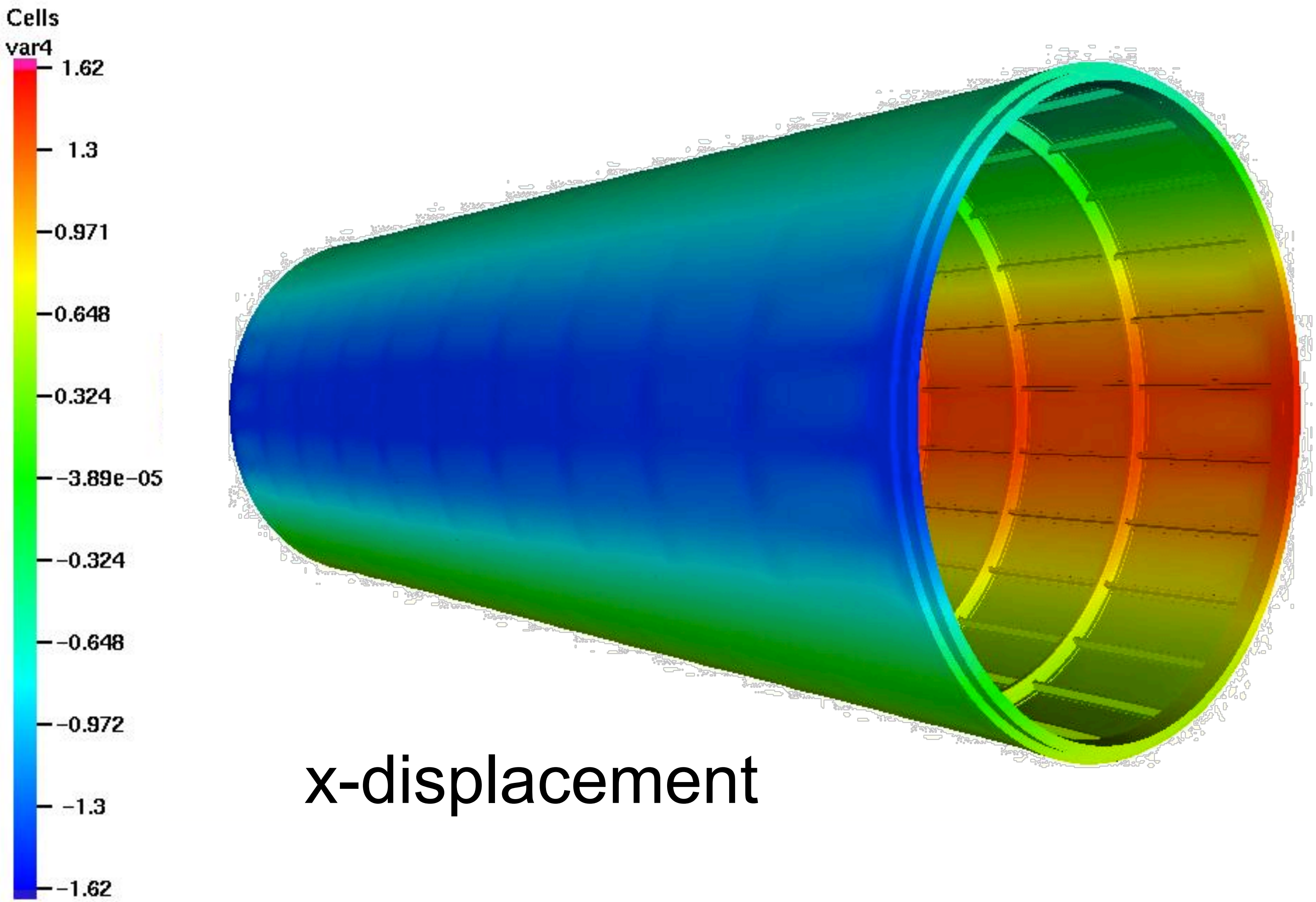


Mesh 2



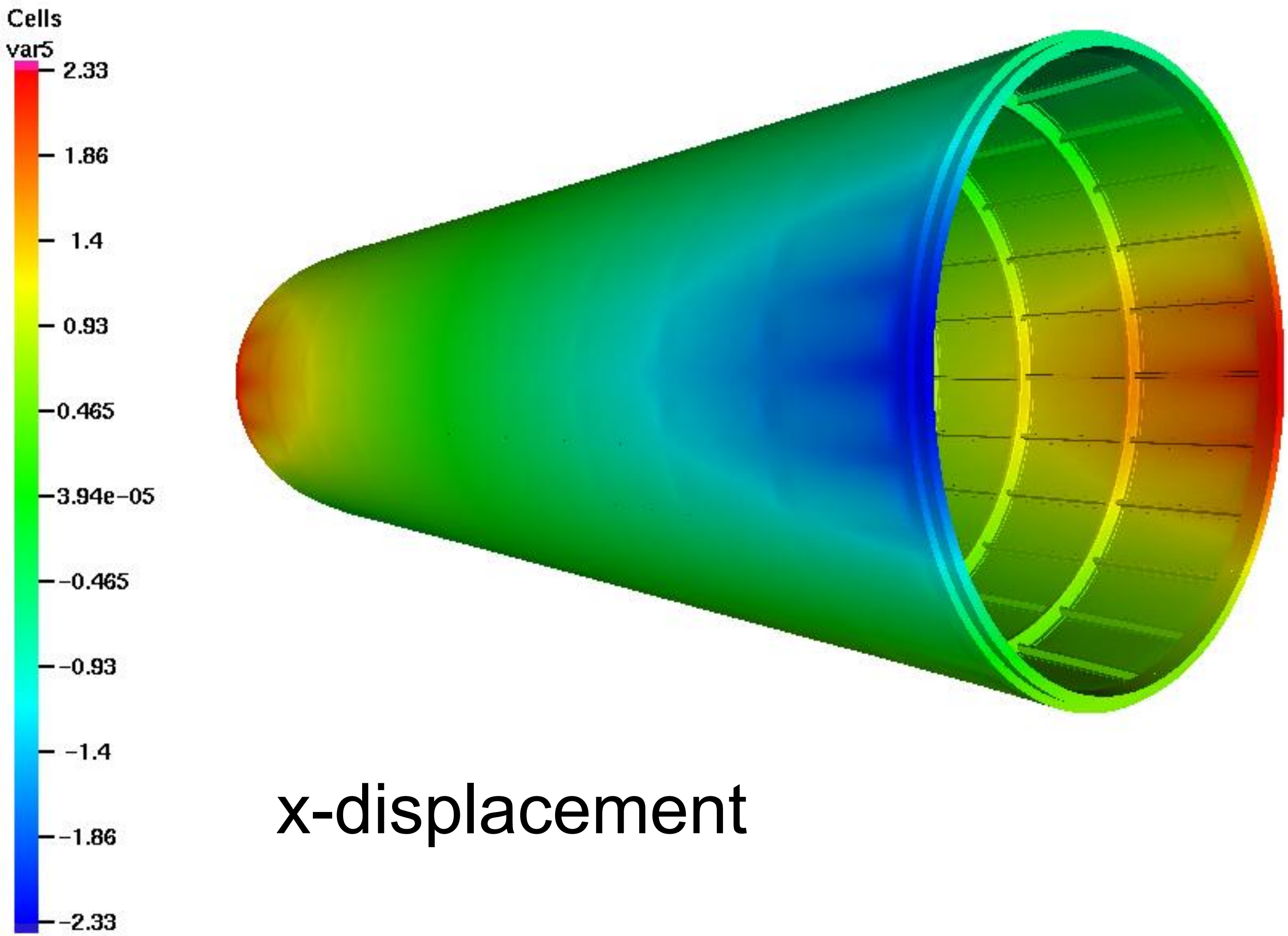
Mesh 3

First Rayleigh Mode



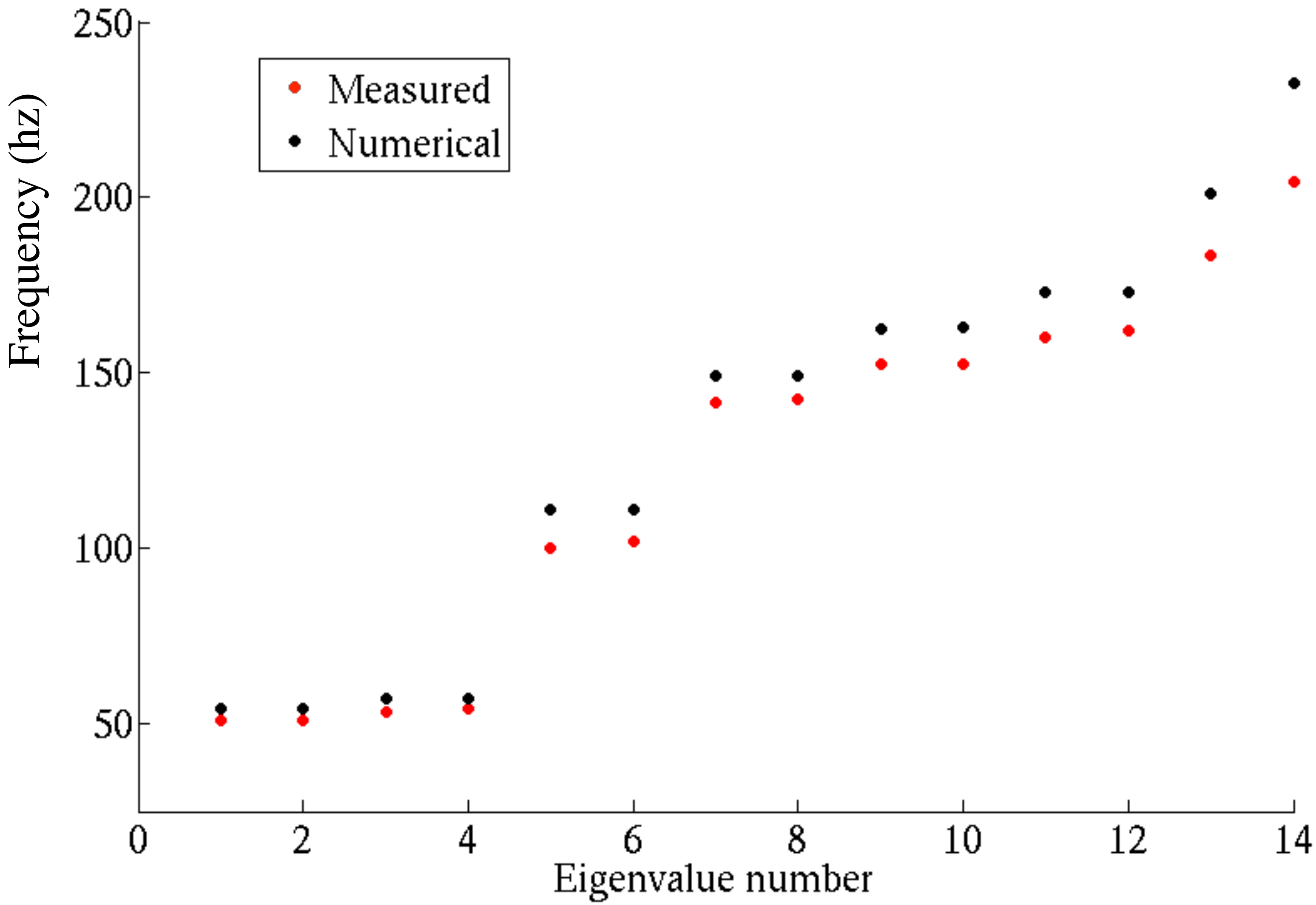
x-displacement

First Love Mode



x-displacement

ATC Frame and Skin



Vibration of a Finite Elastic Rod with Fixed Ends

Problem:

$$\begin{cases} u_{,xx} + \omega^2 u = 0 \quad \text{for} \quad x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

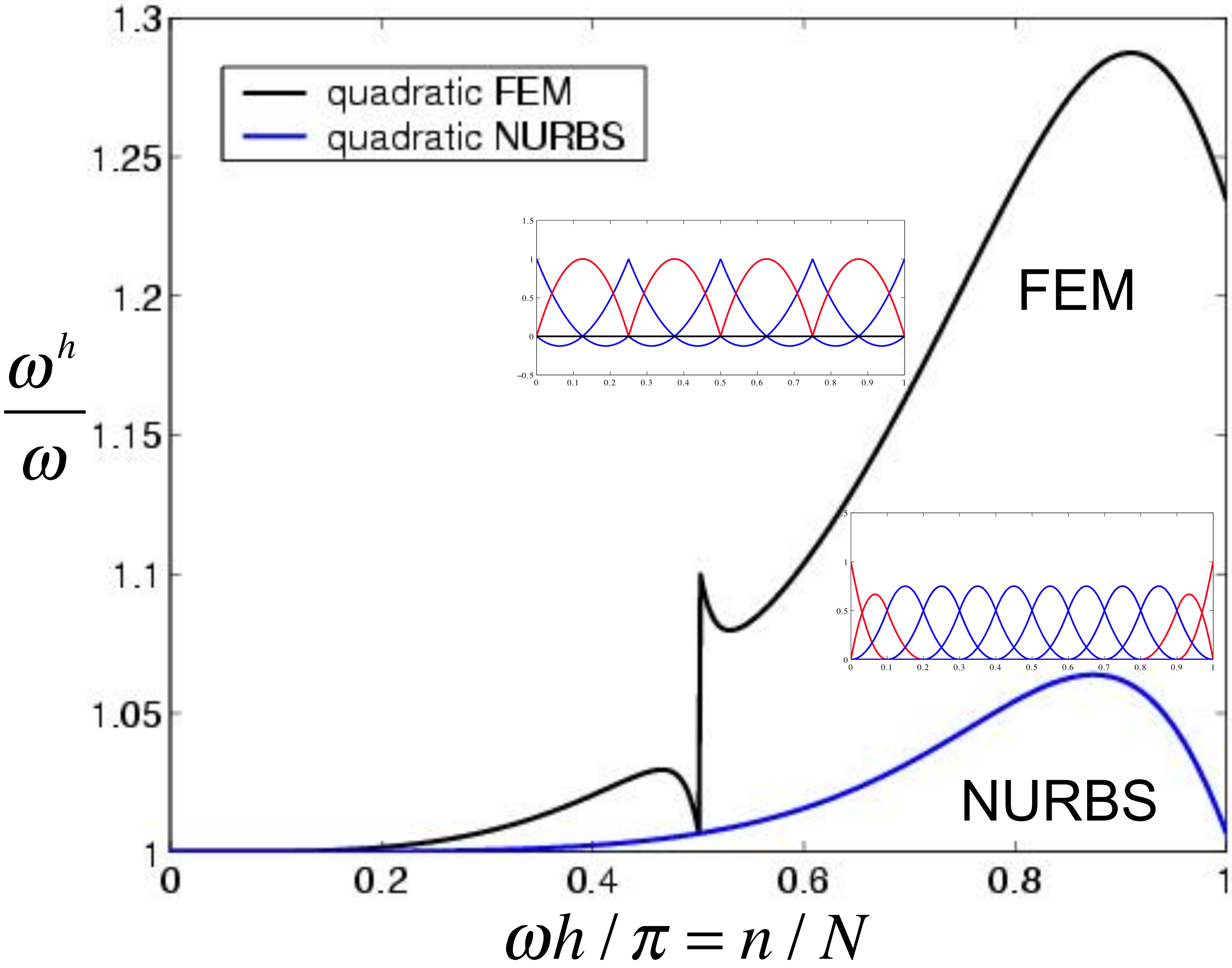
Natural frequencies:

$$\omega_n = n\pi, \quad \text{with} \quad n = 1,2,3,\ldots$$

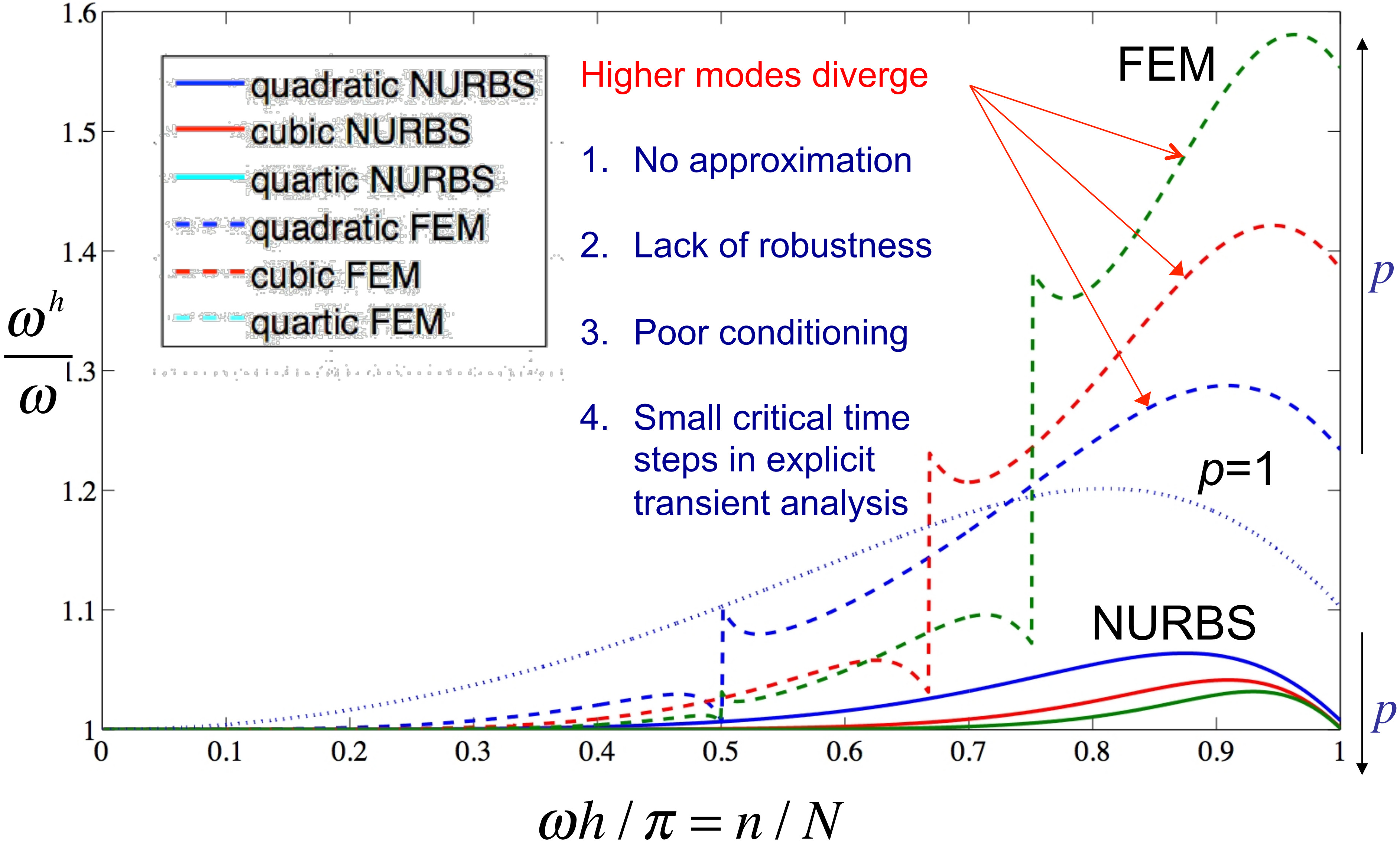
Frequency errors:

$$\omega_n^h / \omega_n$$

Comparison of C^0 FEM and C^{p-1} NURBS Frequency Errors

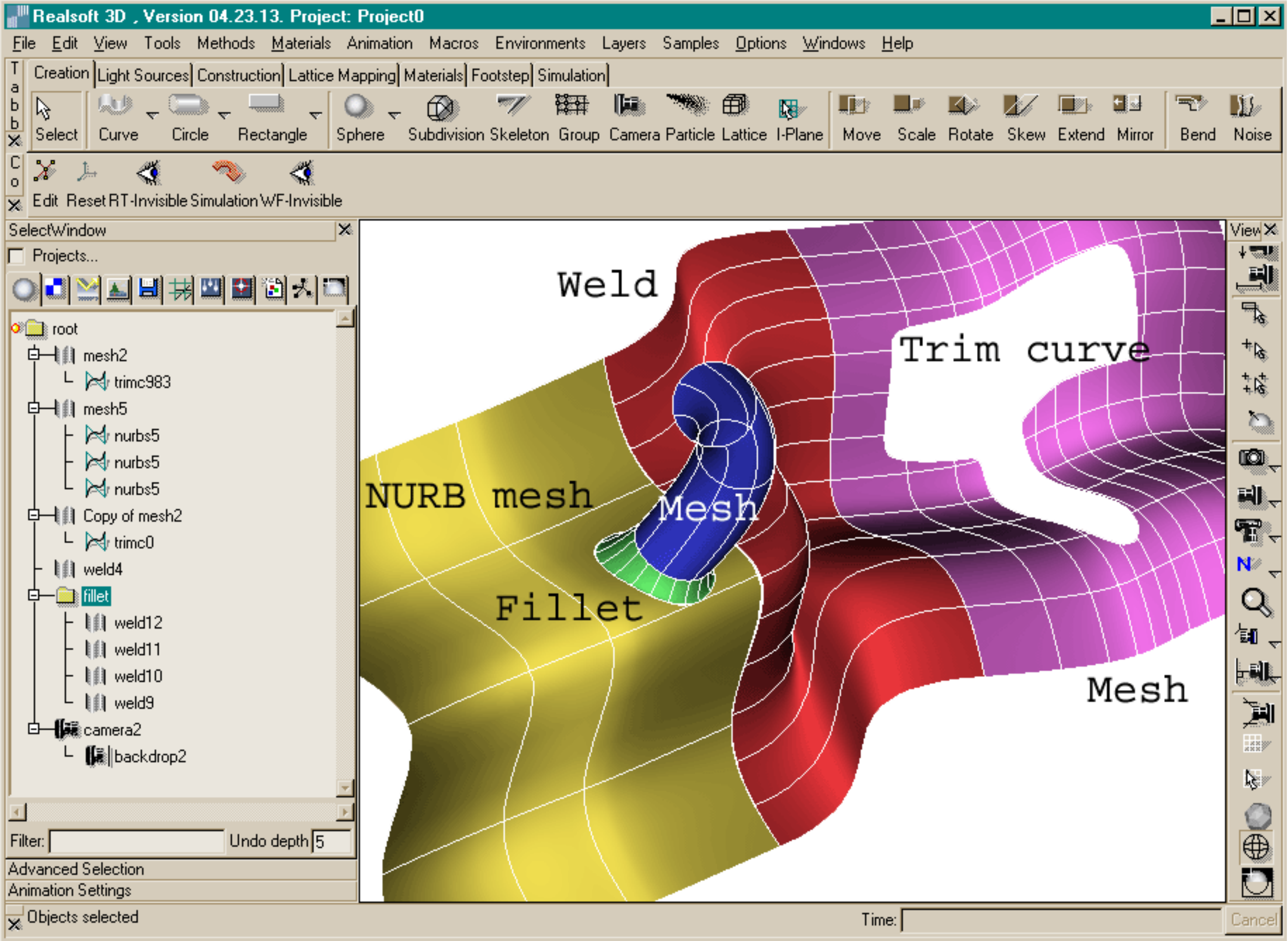


Comparison of C^0 FEM and C^{p-1} NURBS Frequency Errors

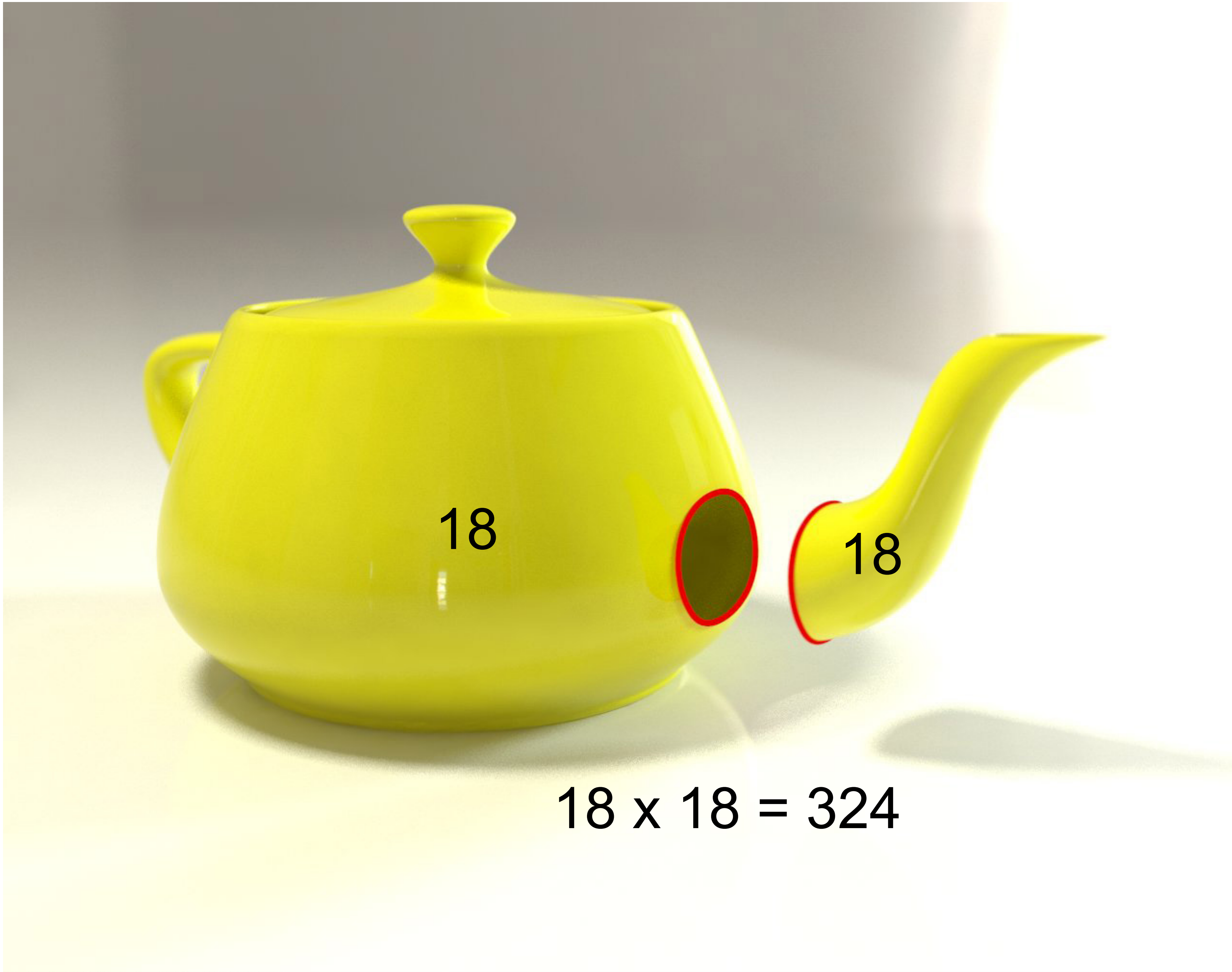


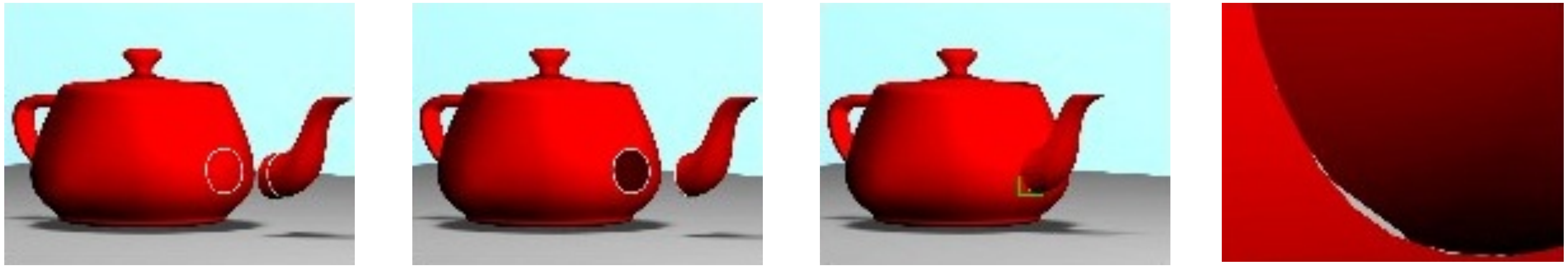
Problems with NURBS-based Engineering Design

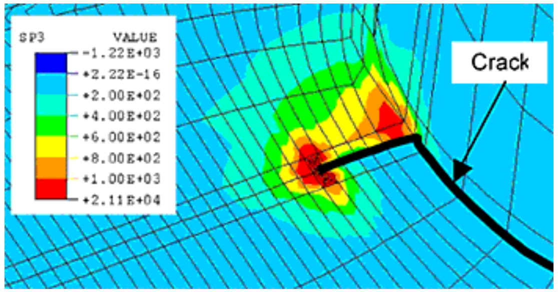
- Water-tight merging of patches
- Trimmed surfaces


Realsoft 3D , Version 04.23.13. Project: Project0
File Edit View Tools Methods Materials Animation Macros Environments Layers Samples Options Windows Help
Creation Light Sources Construction Lattice Mapping Materials Footstep Simulation
Select Curve Circle Rectangle Sphere Subdivision Skeleton Group Camera Particle Lattice I-Plane Move Scale Rotate Skew Extend Mirror Bend Noise
Edit Reset RT-Invisible Simulation WF-Invisible
SelectWindow
Projects...
root
mesh2
trimc983
mesh5
nurbs5
nurbs5
nurbs5
Copy of mesh2
trimc0
weld4
fillet
weld12
weld11
weld10
weld9
camera2
backdrop2
Filter:
Undo depth 5
Advanced Selection
Animation Settings
Objects selected
Time:
Cancel
View
Weld
Trim curve
NURB mesh
Mesh
Fillet
Mesh

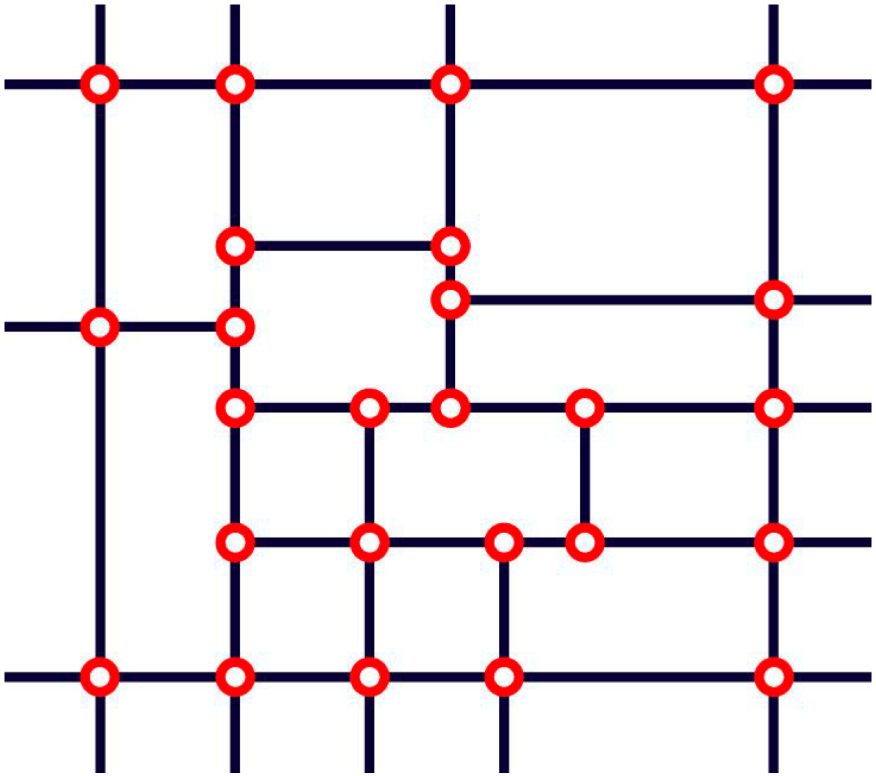
Courtesy of Juan Santocono


18
18
18 x 18 = 324



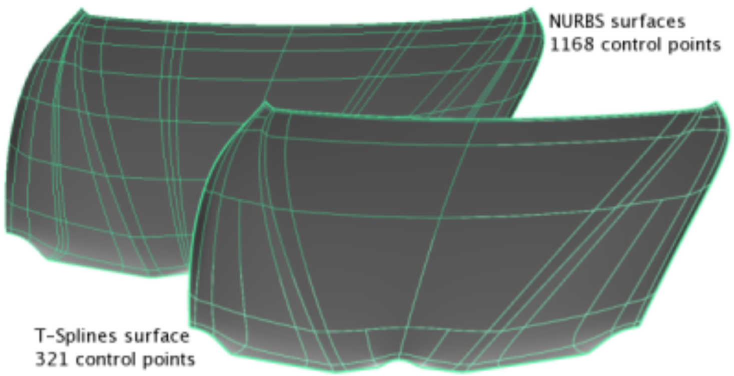

SP3
VALUE
-1.22E+03
+2.22E-16
+2.00E+02
+4.00E+02
+6.00E+02
+8.00E+02
+1.00E+03
+2.11E+04
Crack

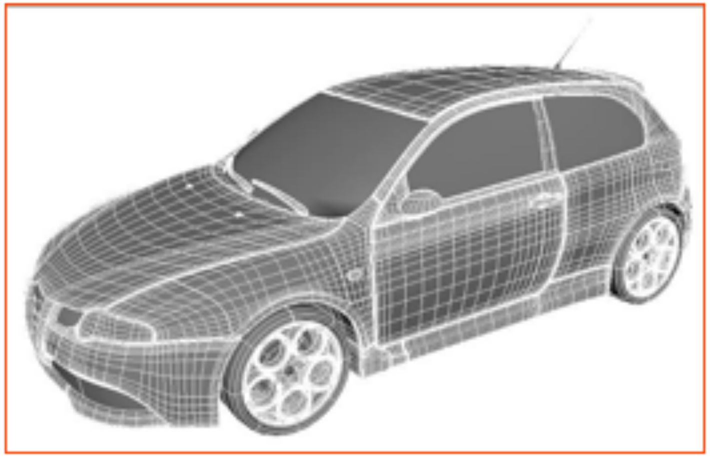
T-splines

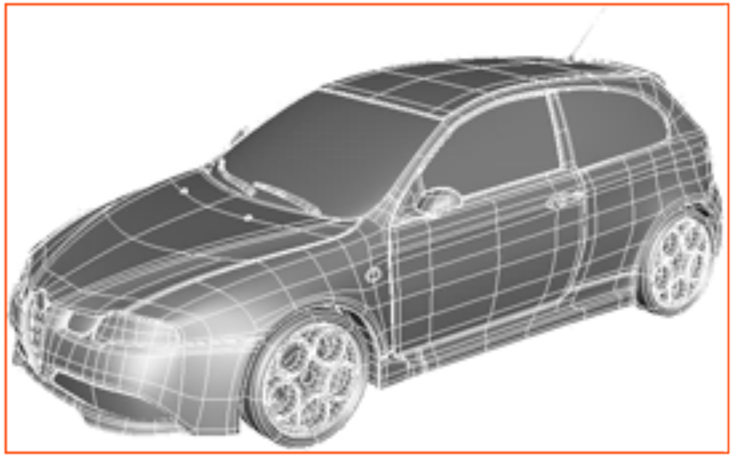


Unstructured NURBS Mesh

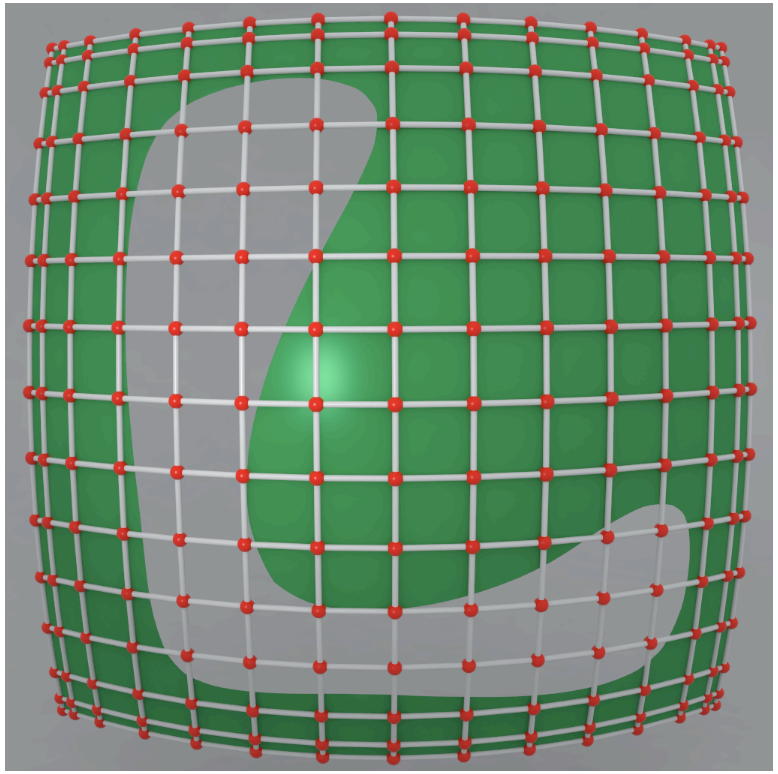
Reduced Number of Control Points



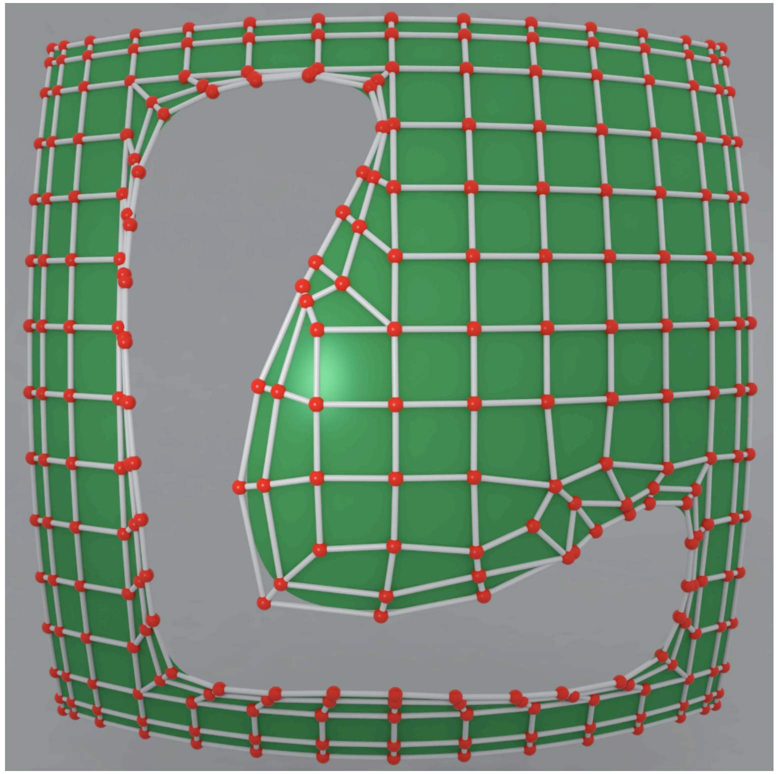




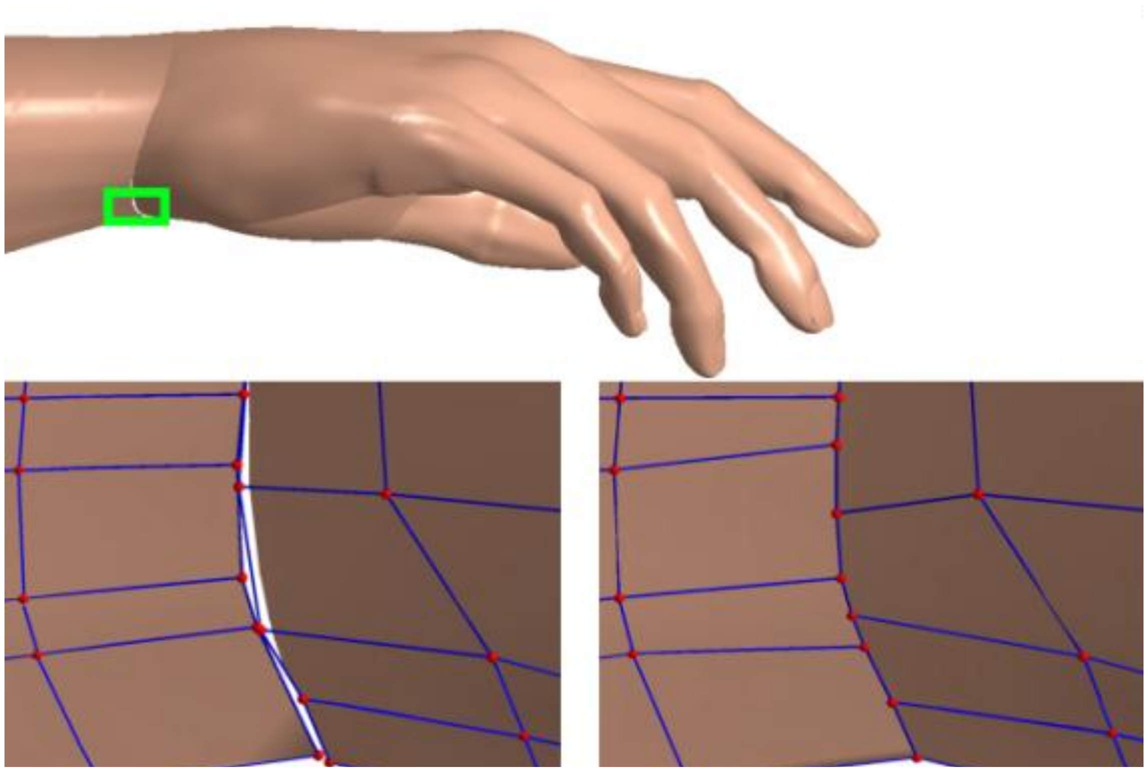
Trimmed NURBS



Untrimmed T-spline



Water-tight merging of patches



Water-tight untrimmed T-spline

Design-through-Analysis*

- Idea:

  Extract surface geometry file from *commercial* CAD modeling software and use it directly in *commercial* FEA software

- Goal:

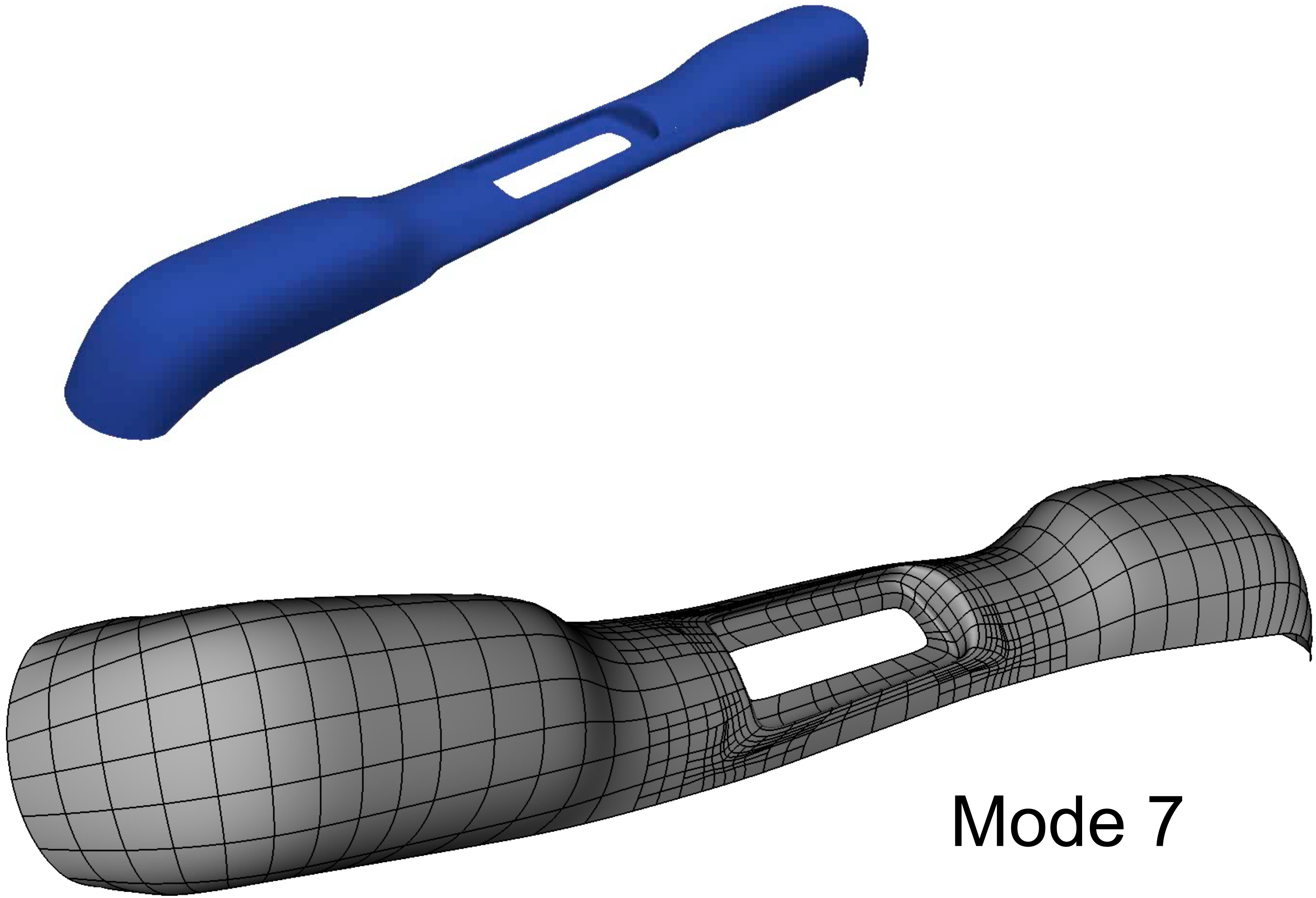
  Bypass mesh generation

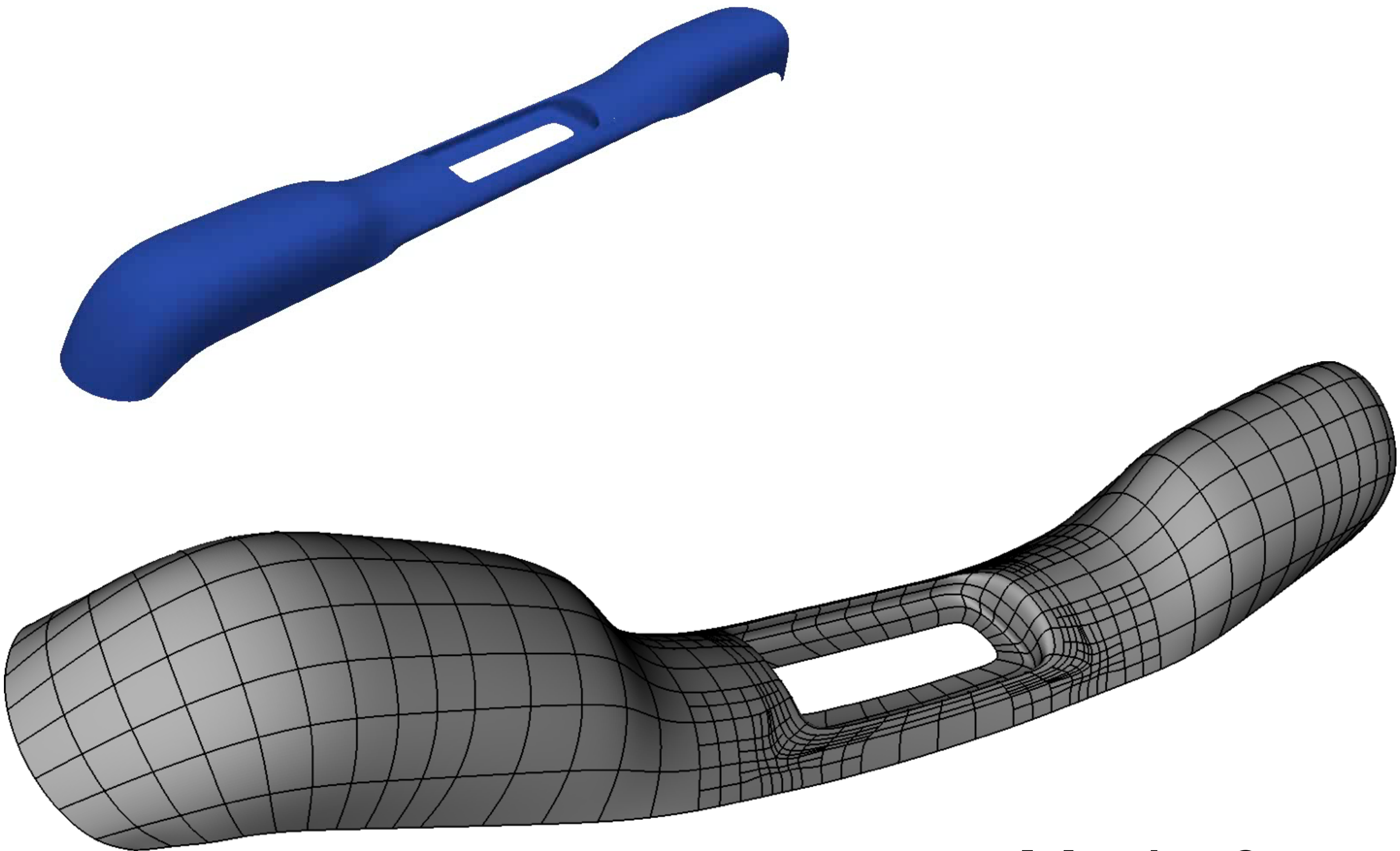
- Test case:

  Import T-spline from Rhino (with T-Spline, Inc. plug-in) surface files directly into LS-DYNA for Reissner-Mindlin shell theory analysis

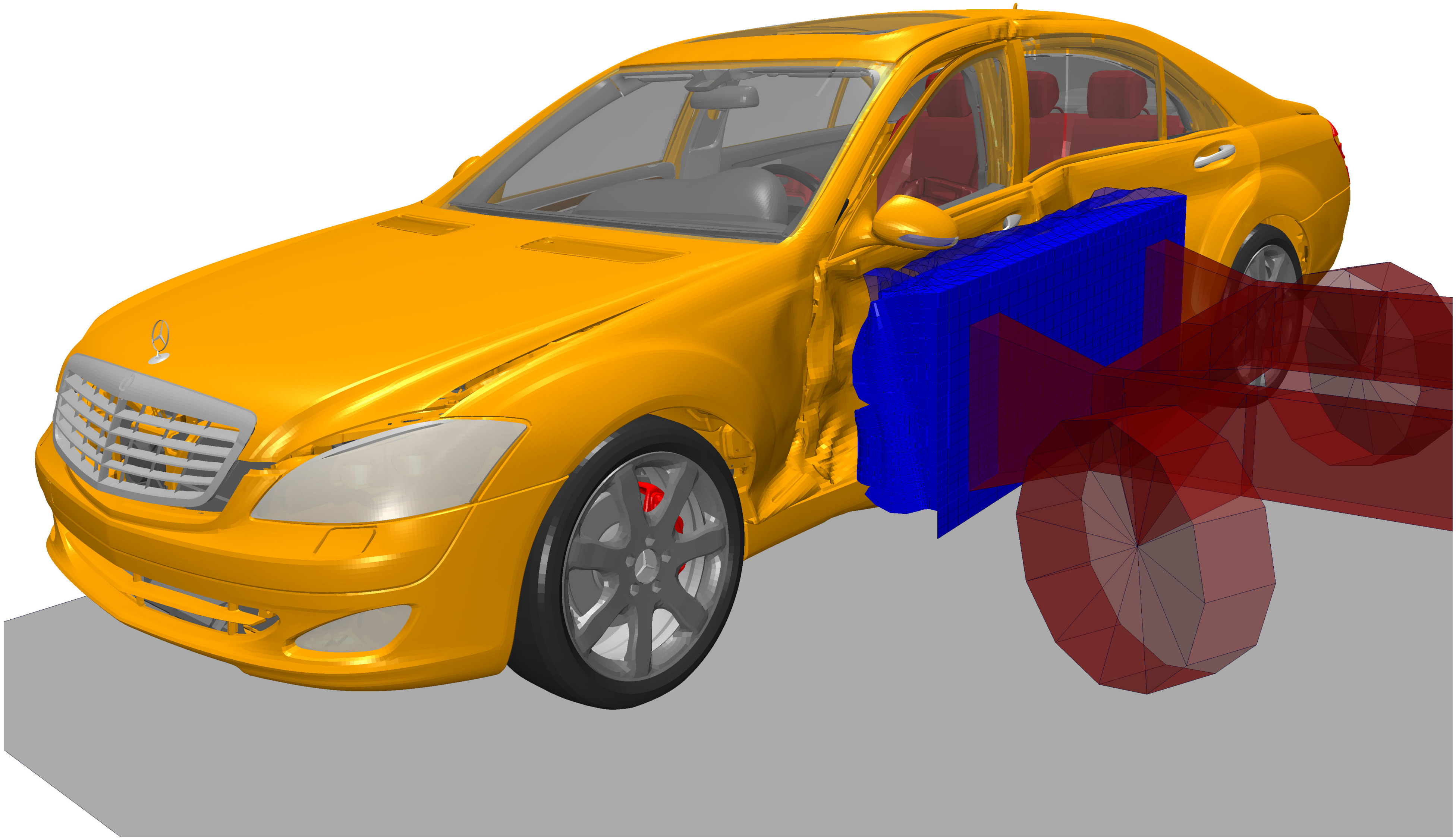
---

*Collaboration with D. Benson


Mode 7



Mode 9



S. Kolling, Mercedez Benz

Nonlocal and Gradient-enhanced Damage-elastic Materials

Constitutive equation

The Caucy stress is assumed related to the infinitesimal strain tensor by the damage-elastic Hooke's law,

$$\sigma_{ij} = \left(1 - \omega\left(\kappa\right)\right) C_{ijkl} \epsilon_{kl}$$

where $\omega \in [0,1]$ is the damage parameter and κ is a history parameter.

Nonlocal Strain Representation

Nonlocality is introduced by defining a nonlocal equivalent strain,

$$\bar{\eta}(x) = \frac{\int_{y \in \Omega} g(x,y)\eta(y)dy}{\int_{y \in \Omega} g(x,y)dy}$$

The weighting function is defined by

$$g(x,y) = \exp\left(-\frac{\| x - y \|^2}{2\ell_c^2} \right)$$

Problem: Dense coefficient matrices

Implicit Gradient Enhancement

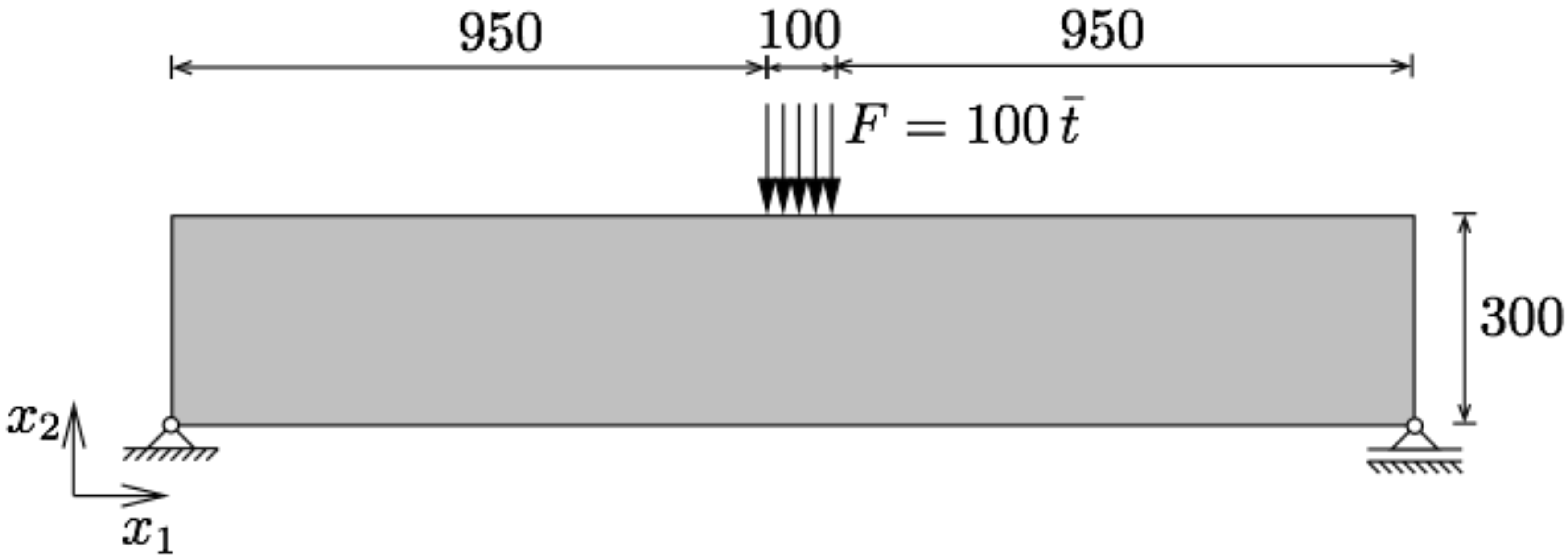
Nonlocal equivalent strain can be approximated by implicit gradient enhancement:

$$\mathcal{L}^d \bar{\eta} \approx \bar{\eta}(x) - \frac{1}{2}\ell_c^2 \frac{\partial^2 \bar{\eta}}{\partial x_i^2}(x) + \frac{1}{8}\ell_c^4 \frac{\partial^4 \bar{\eta}}{\partial x_i^2 \partial x_j^2}(x) - \frac{1}{48}\ell_c^6 \frac{\partial^6 \bar{\eta}}{\partial x_i^2 \partial x_j^2 \partial x_k^2}(x) + \ldots = \eta(x).$$

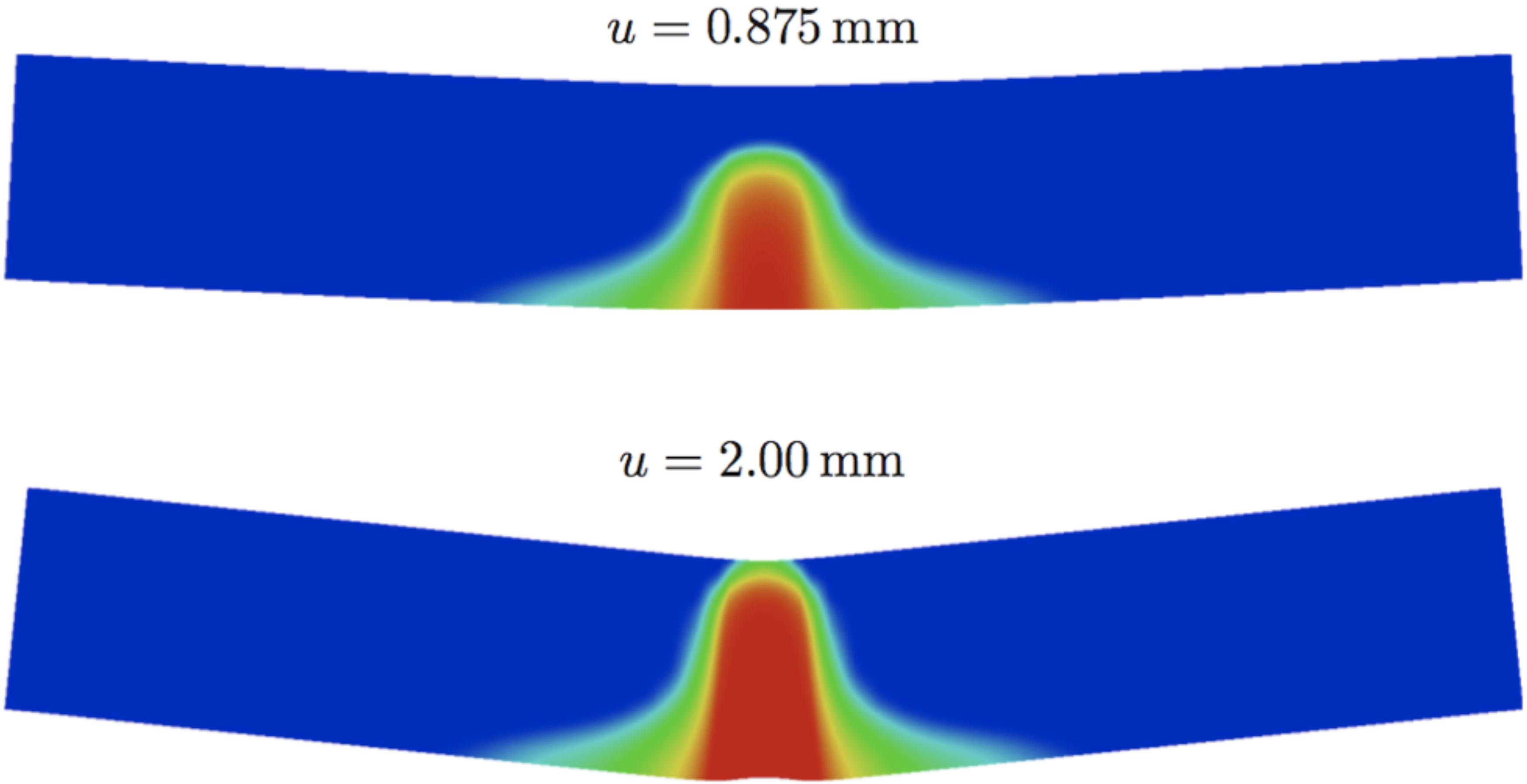
Sixth-order derivatives pose significant implementational problems for C^0-continuous finite elements.

Solution: C^2-continuous T-splines.

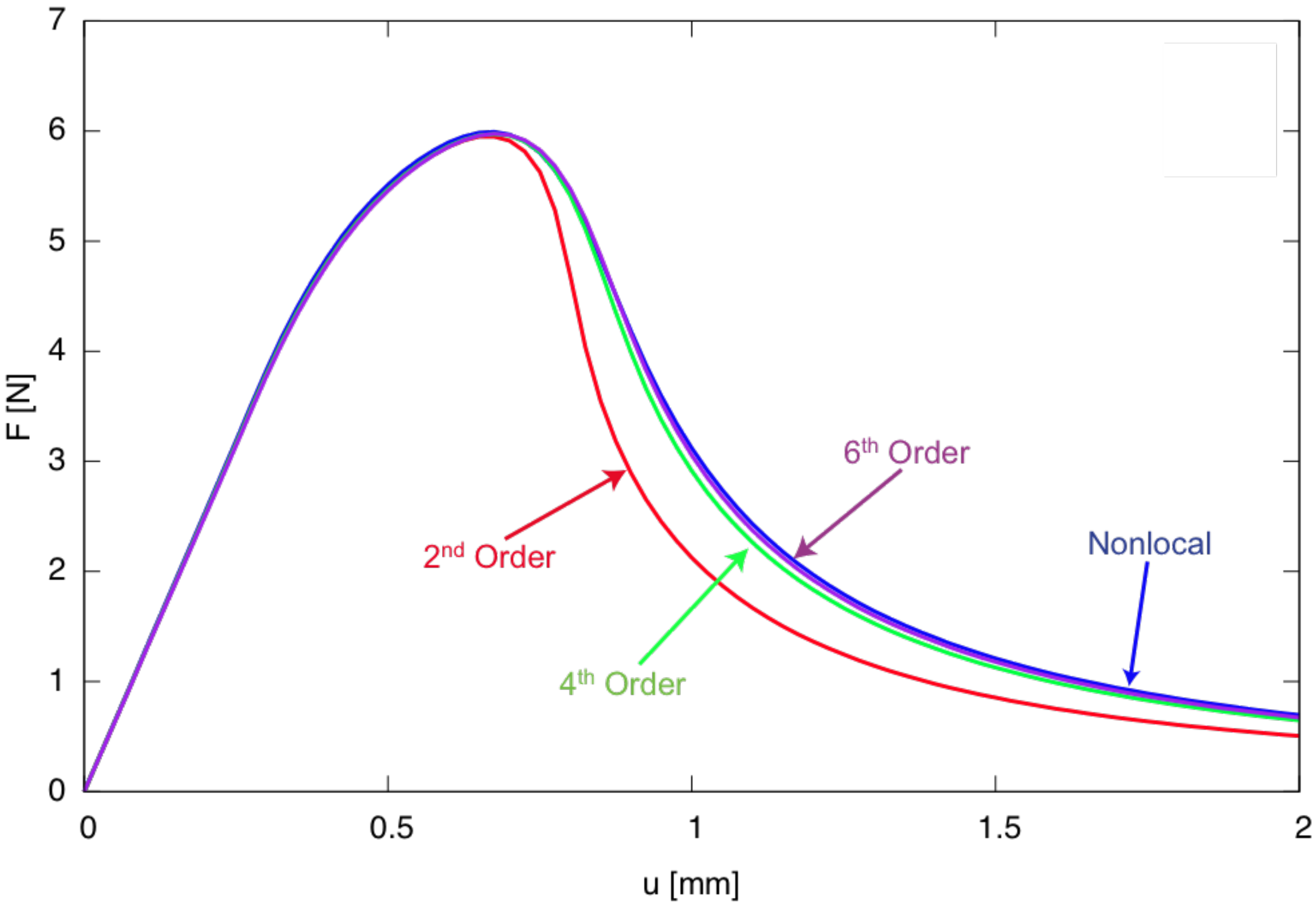
Three Point Bending Problem



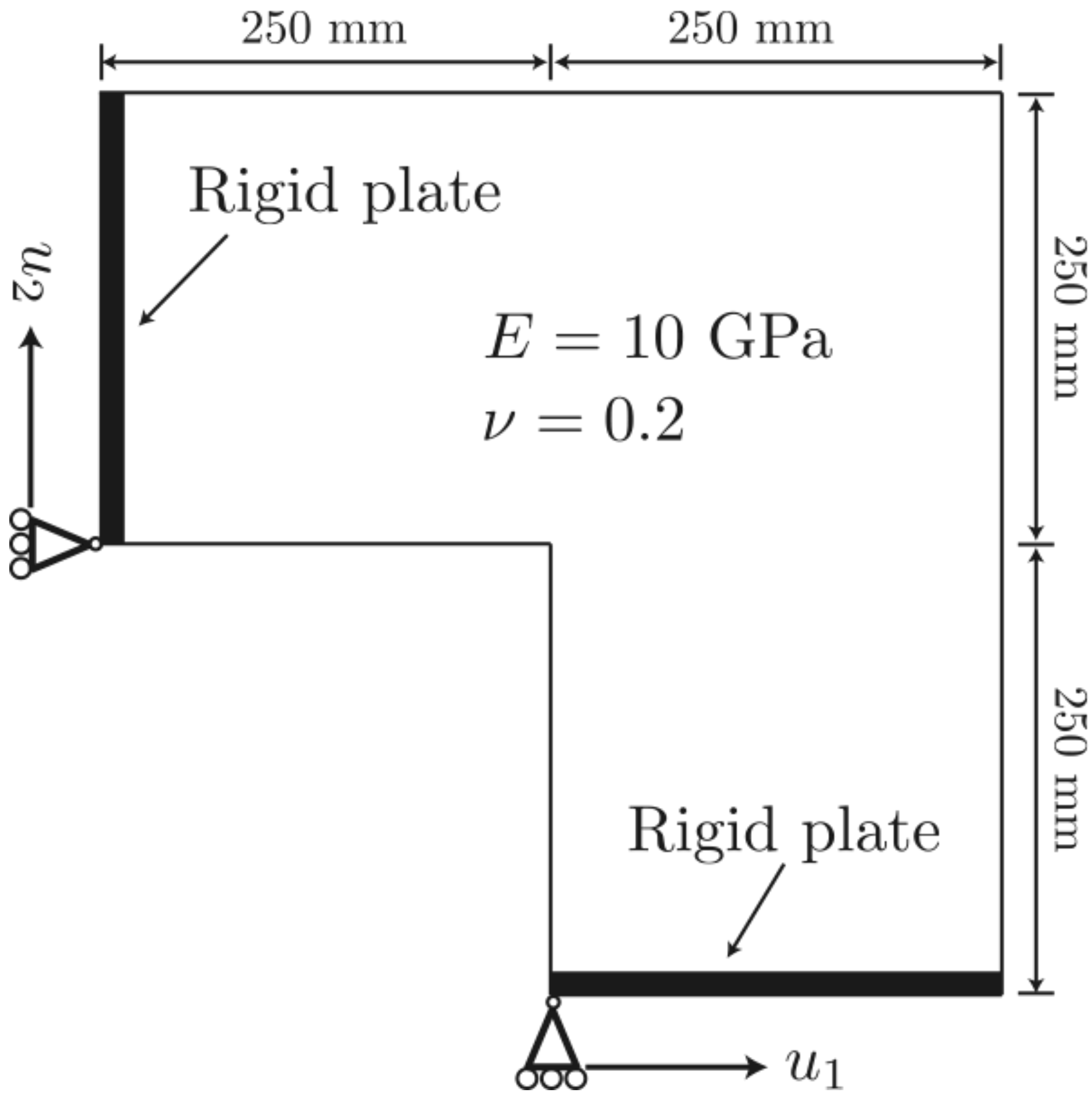
Three Point Bending



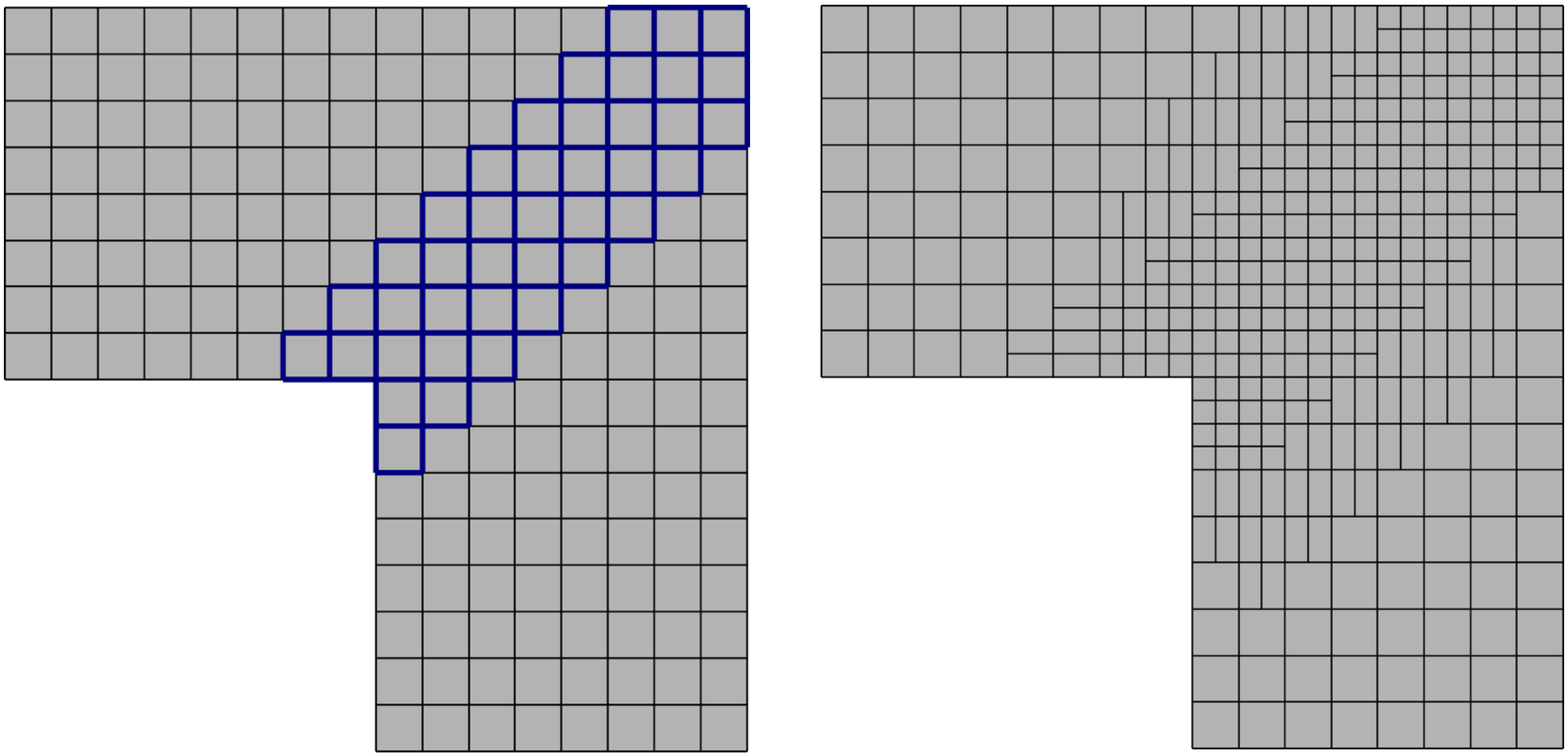
Three Point Bending



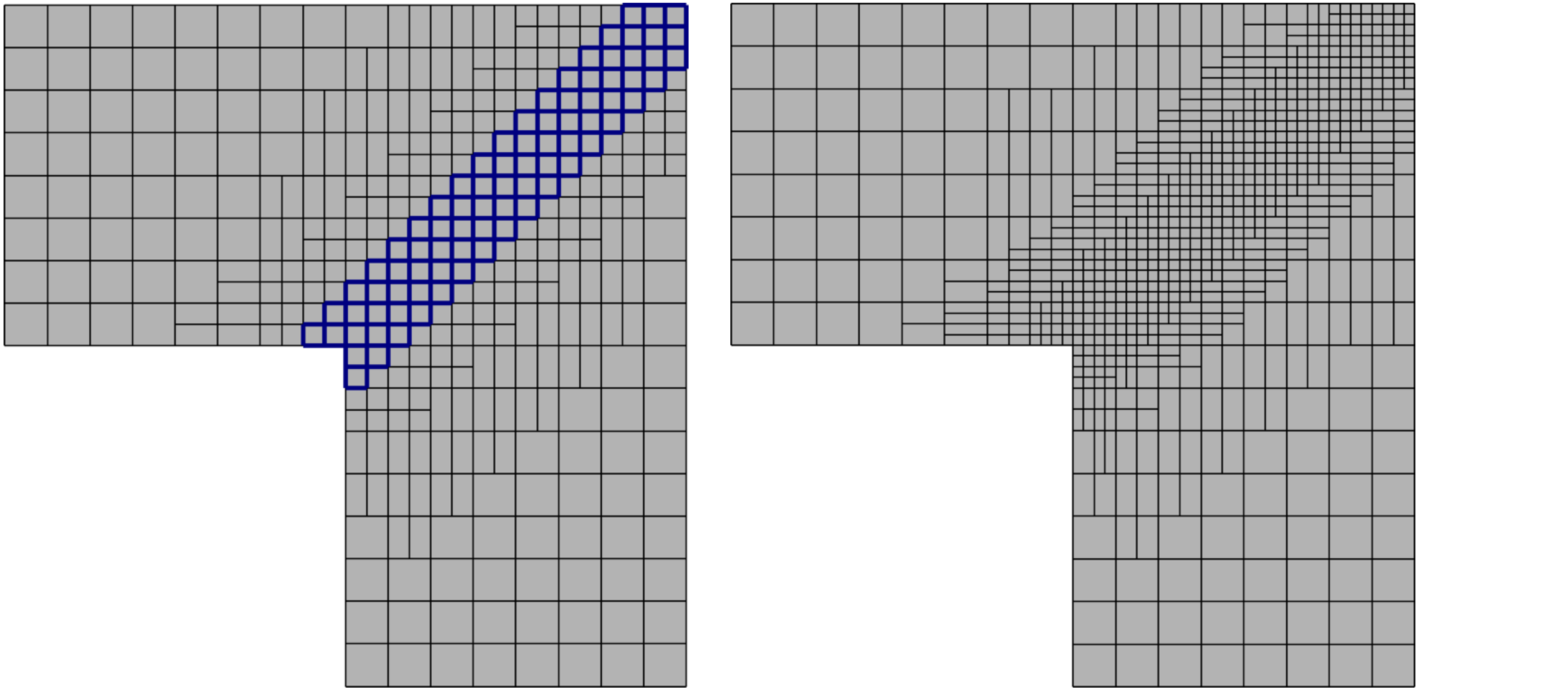
Local Refinement



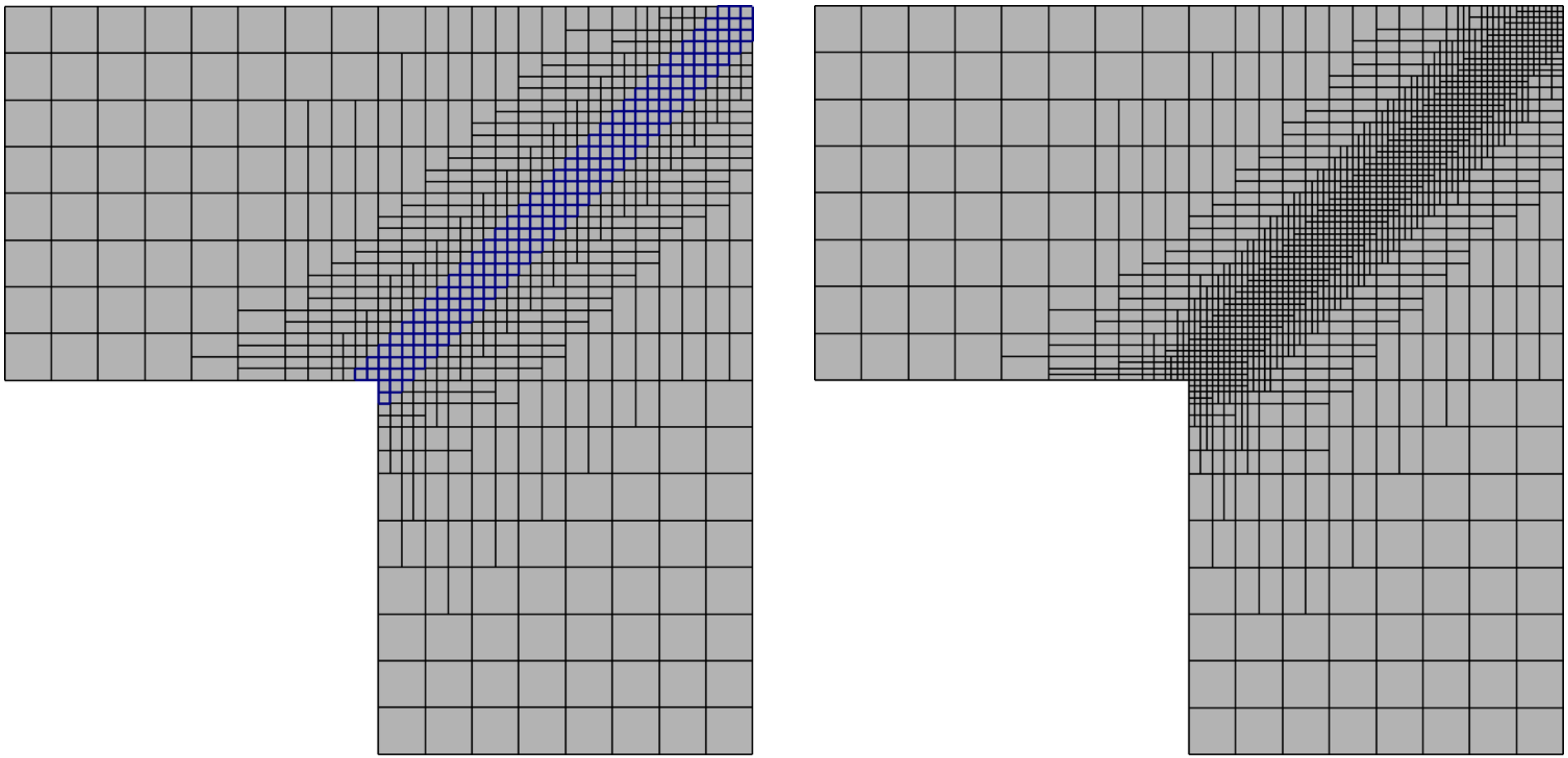
First Refinement

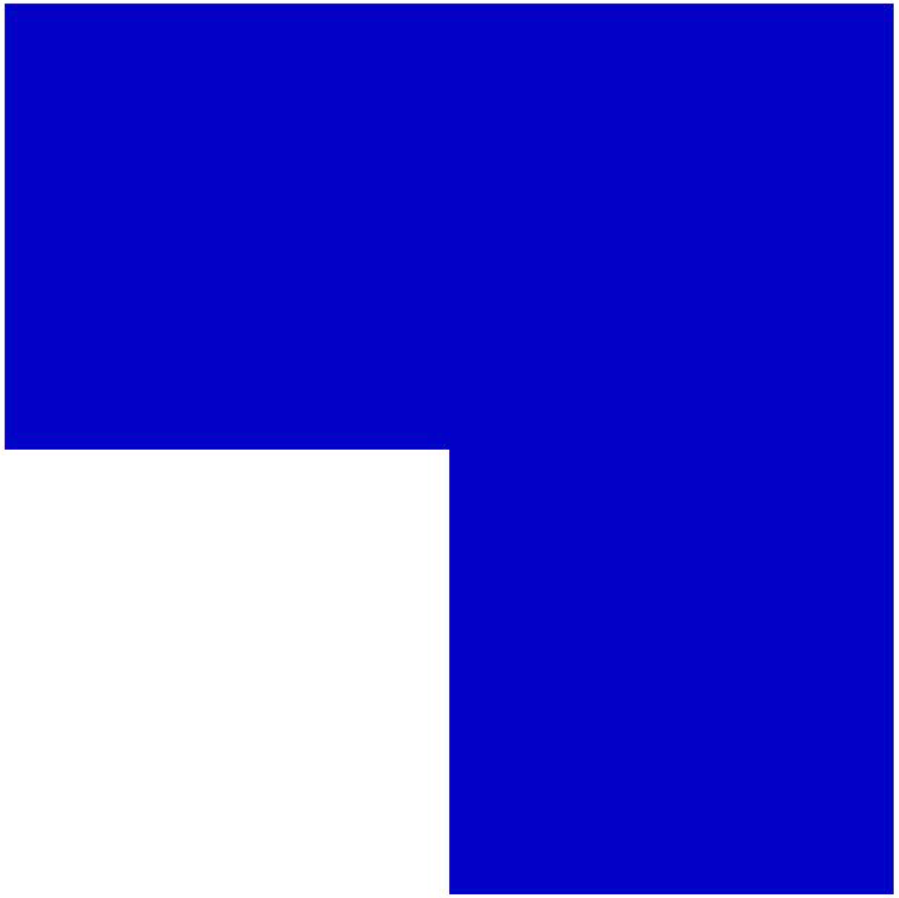


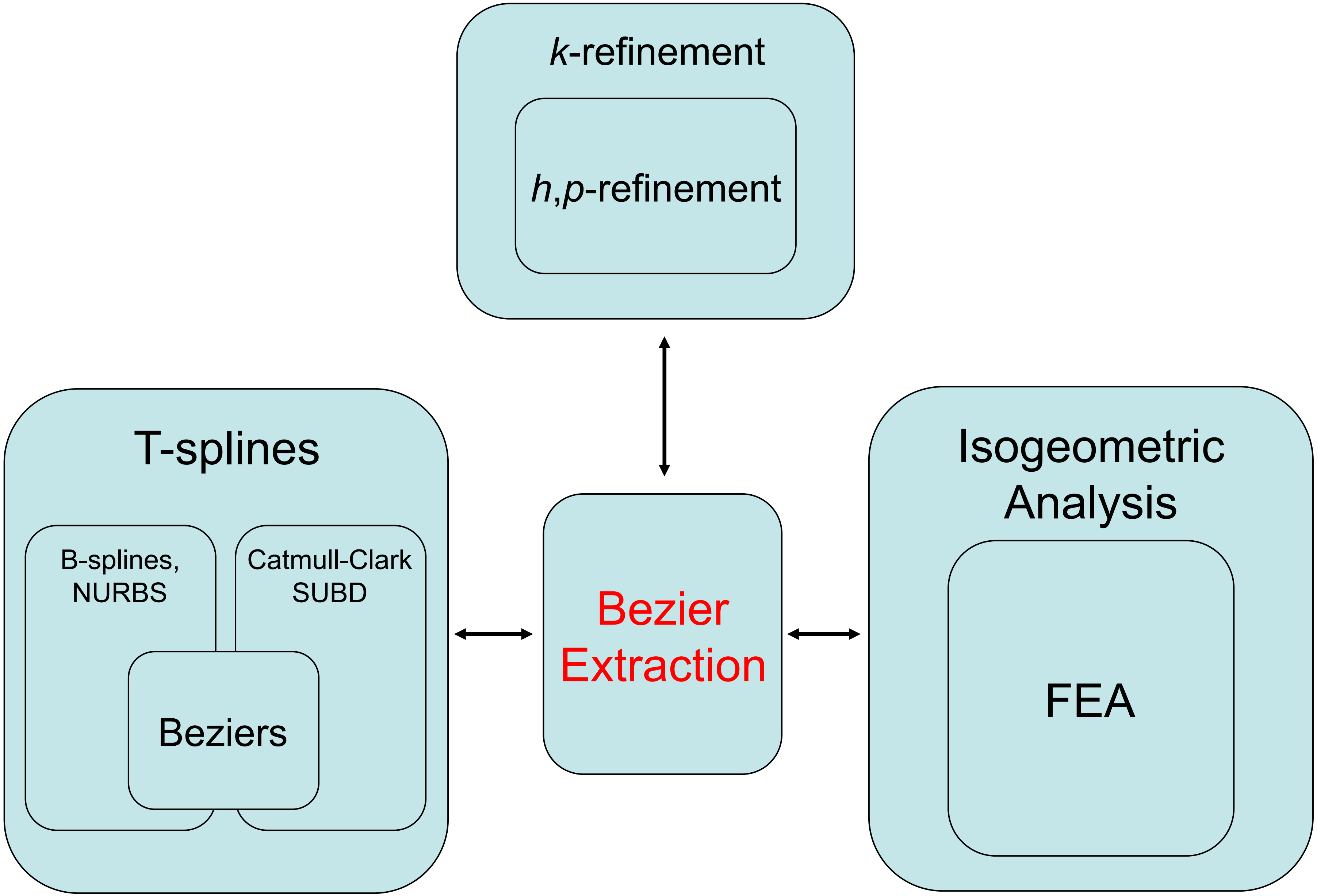
Second Refinement



Third Refinement

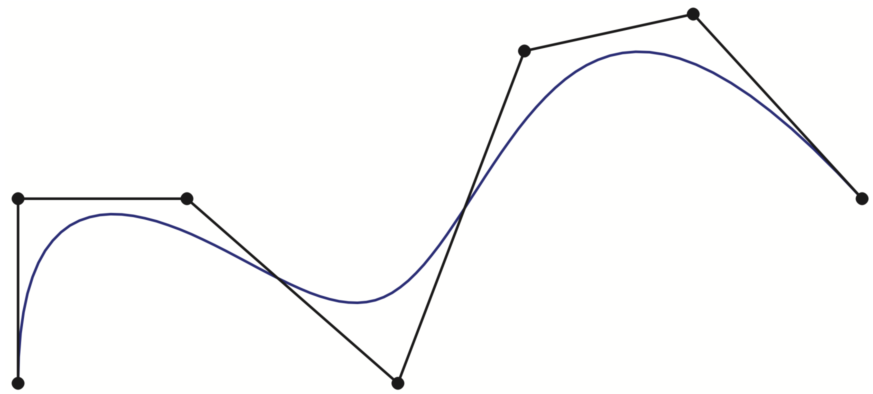


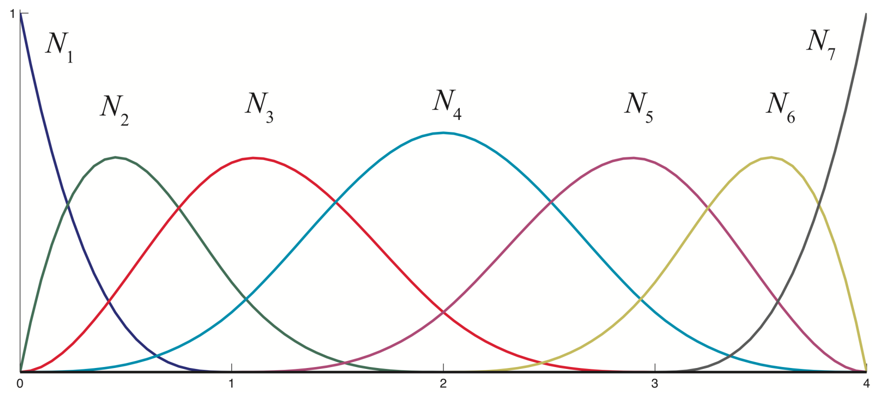




k-refinement
h,*p*-refinement
T-splines
B-splines,
NURBS
Catmull-Clark
SUBD
Beziers
Bezier
Extraction
Isogeometric
Analysis
FEA

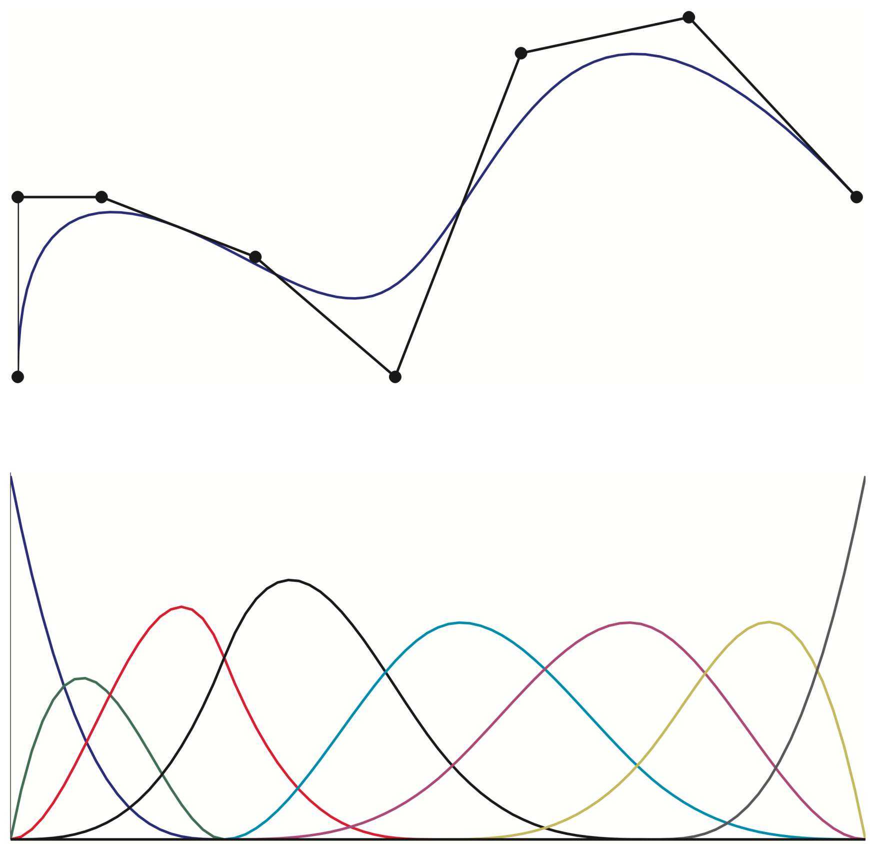
Bezier Decomposition: Repeated Knot Insertion





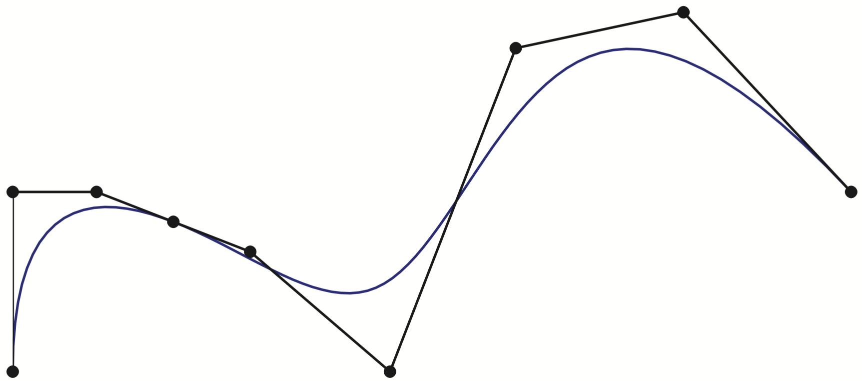
$$\Xi = \{0,0,0,0,1,2,3,4,4,4,4\}$$

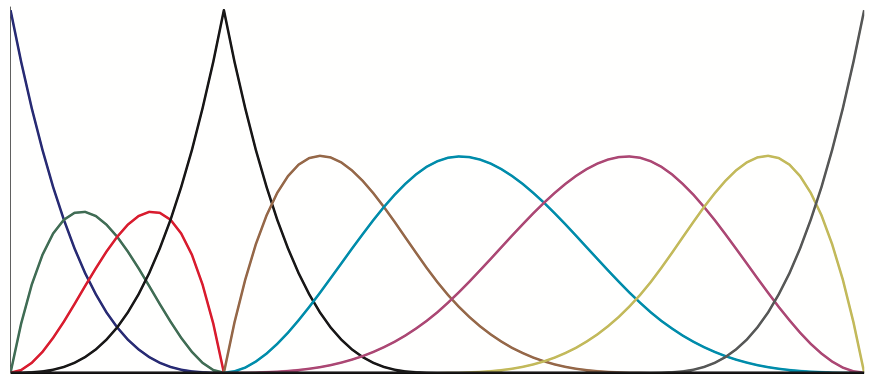
Bezier Decomposition: Repeated Knot Insertion



$$\Xi = \{0,0,0,0,1,1,2,3,4,4,4,4\}$$

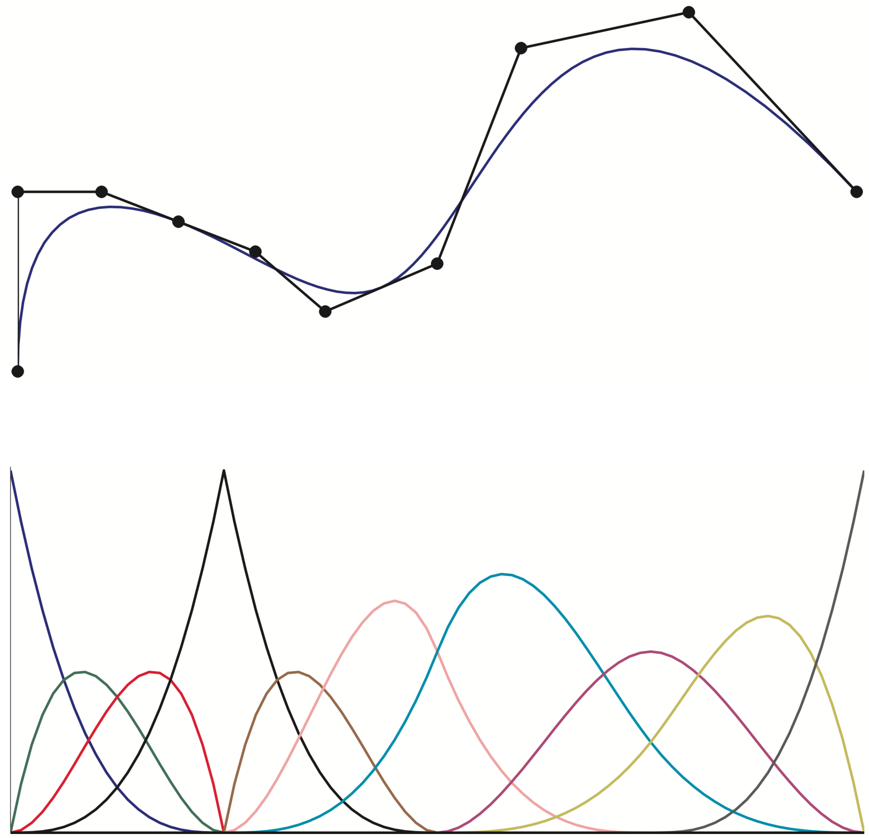
Bezier Decomposition: Repeated Knot Insertion





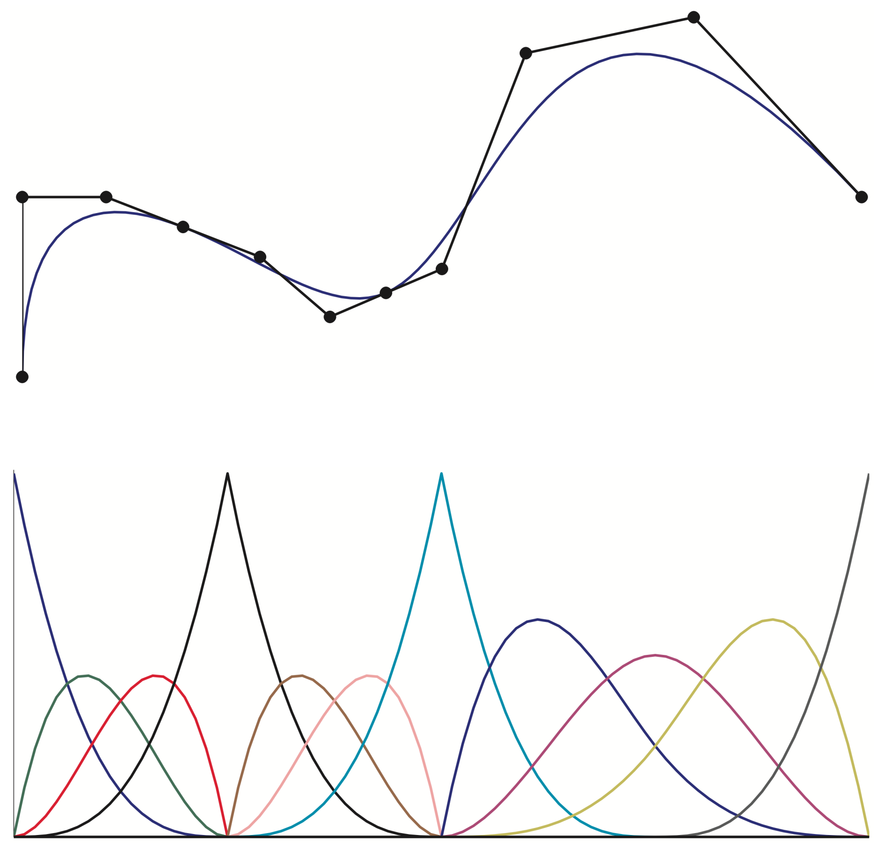
$$\Xi = \{0,0,0,0,1,1,1,2,3,4,4,4,4\}$$

Bezier Decomposition: Repeated Knot Insertion



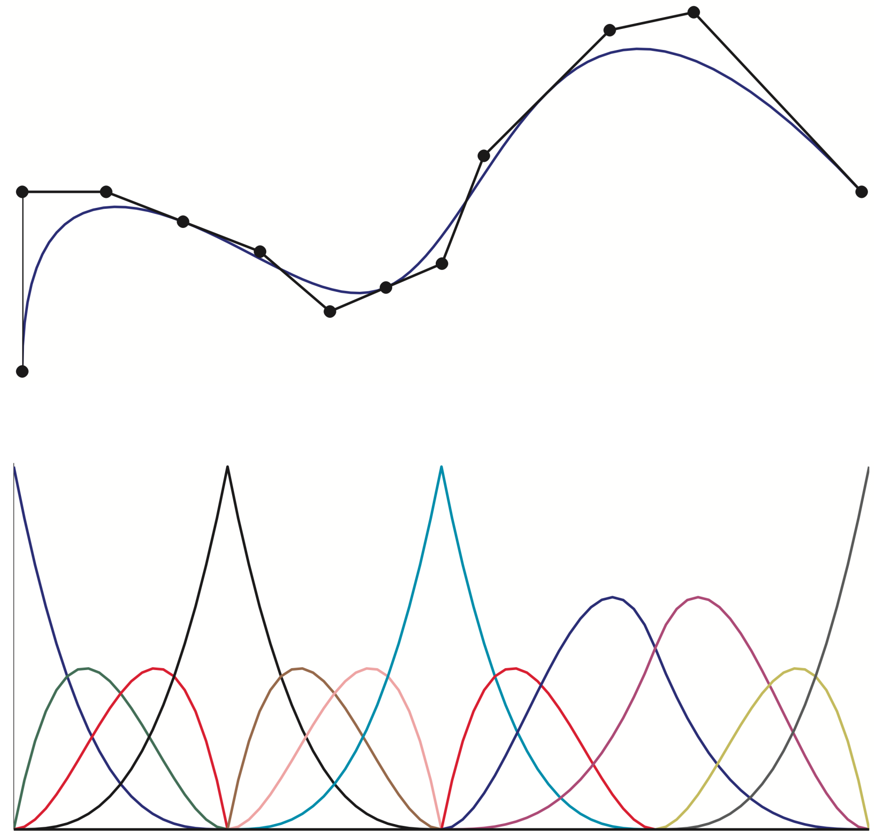
$$\Xi = \{0,0,0,0,1,1,1,2,2,3,4,4,4,4\}$$

Bezier Decomposition: Repeated Knot Insertion



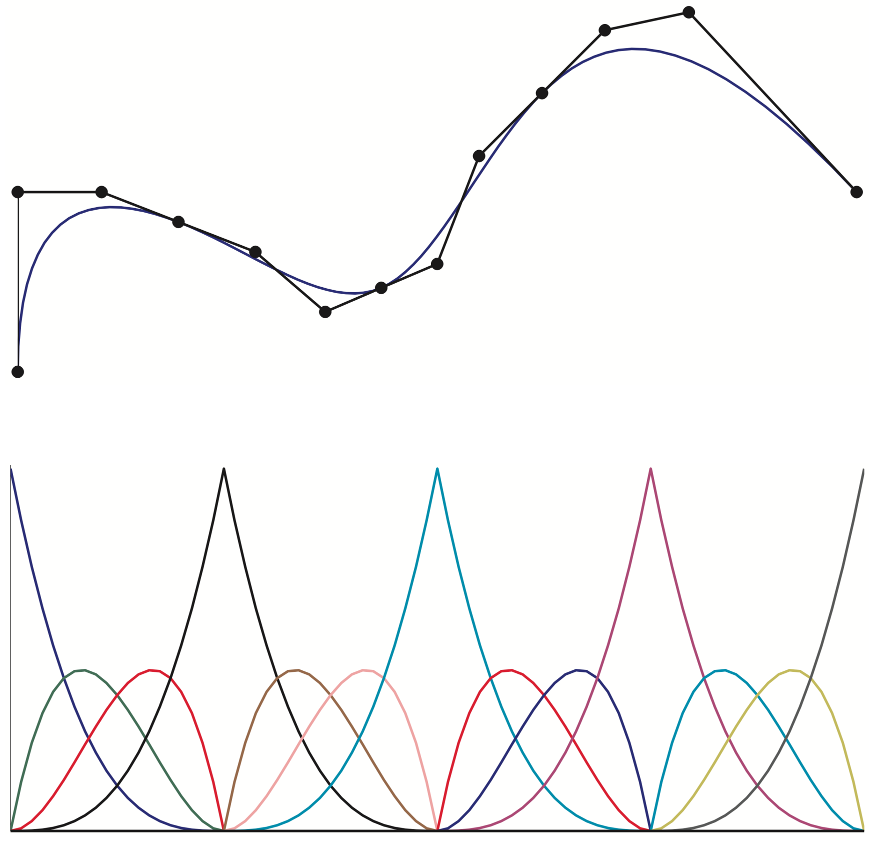
$$\Xi = \{0,0,0,0,1,1,1,2,2,2,3,4,4,4,4\}$$

Bezier Decomposition: Repeated Knot Insertion



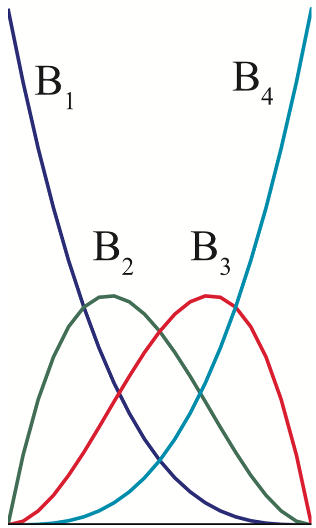
$$\Xi = \{0,0,0,0,1,1,1,2,2,2,3,3,4,4,4,4\}$$

Bezier Decomposition: Repeated Knot Insertion

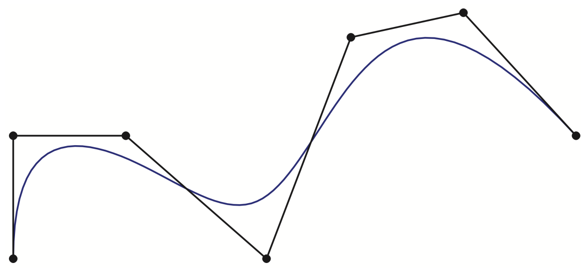


$$\Xi = \{0,0,0,0,1,1,1,2,2,2,3,3,3,4,4,4,4\}$$

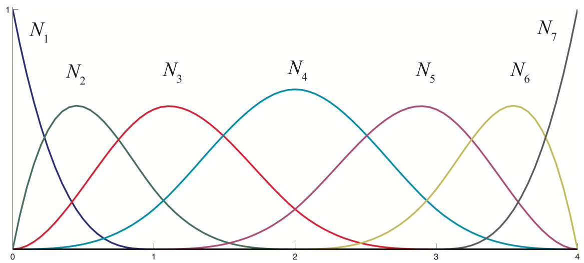
Cubic Bezier Element

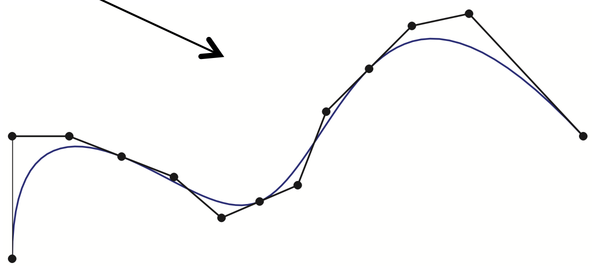


Bezier Decomposition

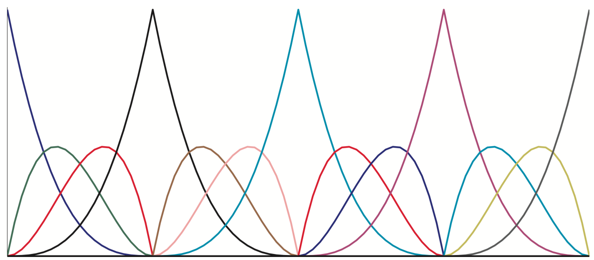


$$\mathbf{P}^b = \mathbf{C}^T \mathbf{P}$$

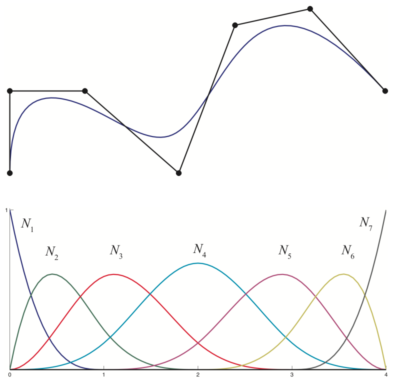




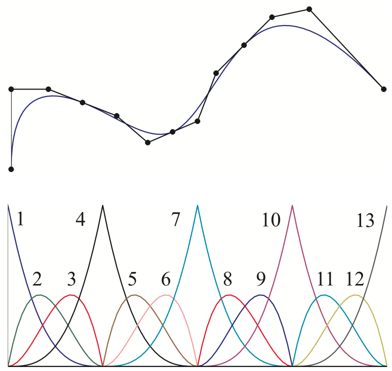
$$\mathbf{N} = \mathbf{C}\mathbf{B}$$



Localizing the Extraction Operator



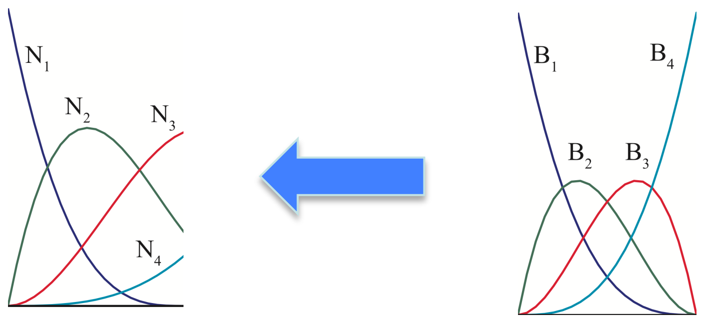
NURBS



Bezier

$$
\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \end{Bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{7}{12} & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{7}{12} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \\ B_{10} \\ B_{11} \\ B_{12} \\ B_{13} \end{Bmatrix}
$$

Localizing the Extraction Operator



NURBS Bezier

In practice, only the local extraction operators are computed.

$$
\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \end{Bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{7}{12} & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{7}{12} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \\ B_9 \\ B_{10} \\ B_{11} \\ B_{12} \\ B_{13} \end{Bmatrix}
$$

Bezier Extraction and the Finite Element Framework

Given f, find $u^h \in \mathcal{S}^h$,

such that, for all $w^h \in \mathcal{V}^h$,

$a(w^h, u^h) = (w^h, f)$

$$w^h = \sum_{A=1}^{n} c_A R_A$$

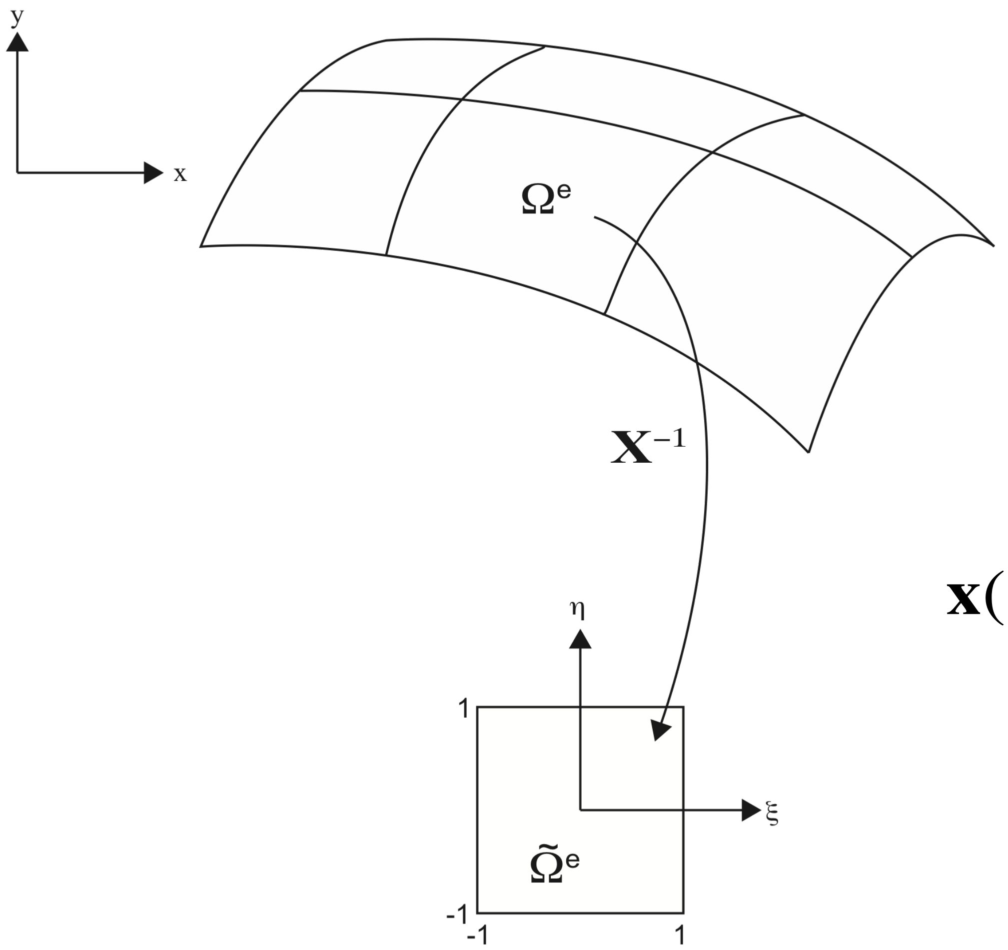
$$u^h = \sum_{B=1}^{n} d_B R_B$$

Matrix Problem:

$$\mathbf{Kd} = \mathbf{F}$$

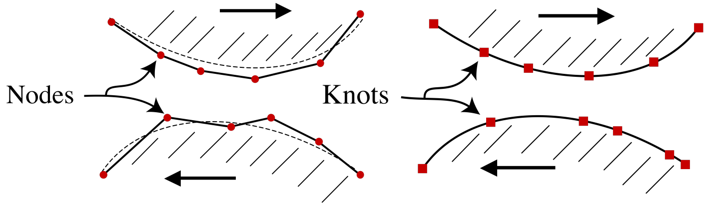
$$\mathbf{K} = [K_{AB}],$$
$$\mathbf{F} = \{F_A\},$$
$$\mathbf{d} = \{d_B\},$$
$$K_{AB} = a(R_A, R_B),$$
$$F_A = (R_A, f)$$

Element Shape Function Routine: Bezier Element Perspective

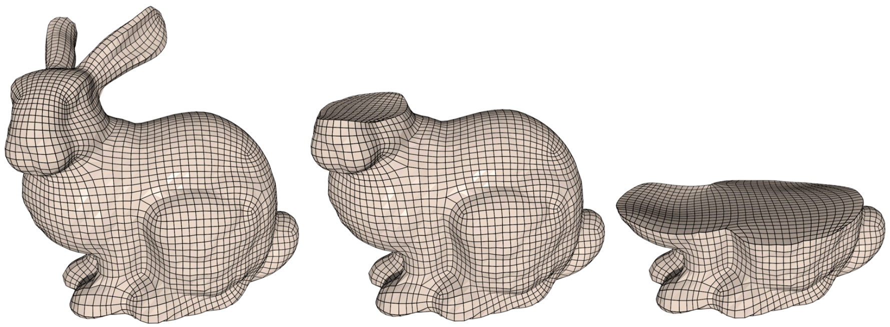


$$\mathbf{x}(\xi) = \sum_{a=1}^{n_{en}} \mathbf{P}_a^e R_a^e(\xi)$$

Research Progress:



- Shape and Topology Optimization ☑
- Efficient quadrature and collocation ☑
- Mathematical theory ☑
  - *h*-convergence ☑
  - Kolmogorov *n*-widths ☑
- Various nonlinear structural applications ☑
  - Shells, w/wo rotational DOF ☑
  - Implicit gradient enhanced damage ☑
  - Contact ☑, frictional sliding ☐
- Turbulence and fluid-structure interaction ☑
- Aero- and hydro-acoustics ☑
- Phase-field methods ☑
  - Navier-Stokes-Korteweg equations ☑
  - Crack propagation ☑
- Electromagnetics (Buffa, Sangalli, Vasquez, et al.) ☑
- Efficient mesh refinement algorithms ☑
- Integration of modeling and analysis tools ☑
  - Analysis-suitable surface descriptions ☐
  - Analysis-suitable volume parameterizations from CAD surfaces ☐



(David Gu)

USACM / ICES Thematic Conference: **IGA 2011**

Isogeometric Methods – Integrating Design and Analysis

Institute for Computational Engineering and Sciences (ICES)
University of Texas at Austin
January 13-15, 2011, Save the date!

Organizers: David Benson, Yuri Bazilevs, Thomas J.R. Hughes.