On the Prony Relaxation Function

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Abstract

For solving viscoelastic problems, the constitutive equations involve convolution integrals with relaxation functions. The relaxation function, $G(t)$, is often written in Prony series

$$G(t) = G_0 + \sum_{i=1}^{N} G_i e^{-\beta_i t}$$

The material constants, G_0, G_i, β_i and N are determined from relaxation test data. In this paper we present the determination of these material constants.

Introduction

For solving viscoelastic problems, we deal with convolution integrals. The simplest constitutive equation is the linear integral equation

$$\sigma_{ij} = \int_0^t G_{ijkl}(t-\tau)\frac{\partial \varepsilon_{kl}}{\partial \tau} d\tau \tag{1}$$

The relaxation function may be represented by the Prony series:

$$G(t) = G_0 + \sum_{i=1}^{N} G_i e^{-\beta_i t} \tag{2}$$

The subject of this paper is to determine the material constants G_0, G_1, … , G_N, β_1, … , β_N and N. For simplicity we assume

$$\begin{aligned} &\beta_1 = \beta_1 \\ &\beta_2 = 10\beta_1 \\ &. \\ &. \\ &. \\ &\beta_N = 10^N \beta_1 . \end{aligned} \tag{3}$$

G_0, G_1, … , G_N are then determined from the least-square error minimization of a set of linear algebraic equations. The starting value, β_1, and the number of terms, N, need to be selected by users. In this paper a general guidance for selecting these values is presented.

Data Analysis

The viscoelastic relaxation data for a uniaxial extension test are shown in Figures 1 and 2. Figure 1 shows the short-term test data. Figure 2 shows the long-term test data. In the test the displacement at one end of a uniaxial test specimen is 0.3 inches and the specimen before deformation is 3 inches; thus the strain is 0.1 (10%). The relaxation stress seems to approach a steady state, as shown in Figure 1; however, when we look closely at the log-log plot the steady state is not reached, as shown in Figure 2. For determining the Prony series material constants, the test data has to be further reduced; each data point will be equally spaced in log time, thus equally weighted during data analysis. The selected data are shown in Figure 3.

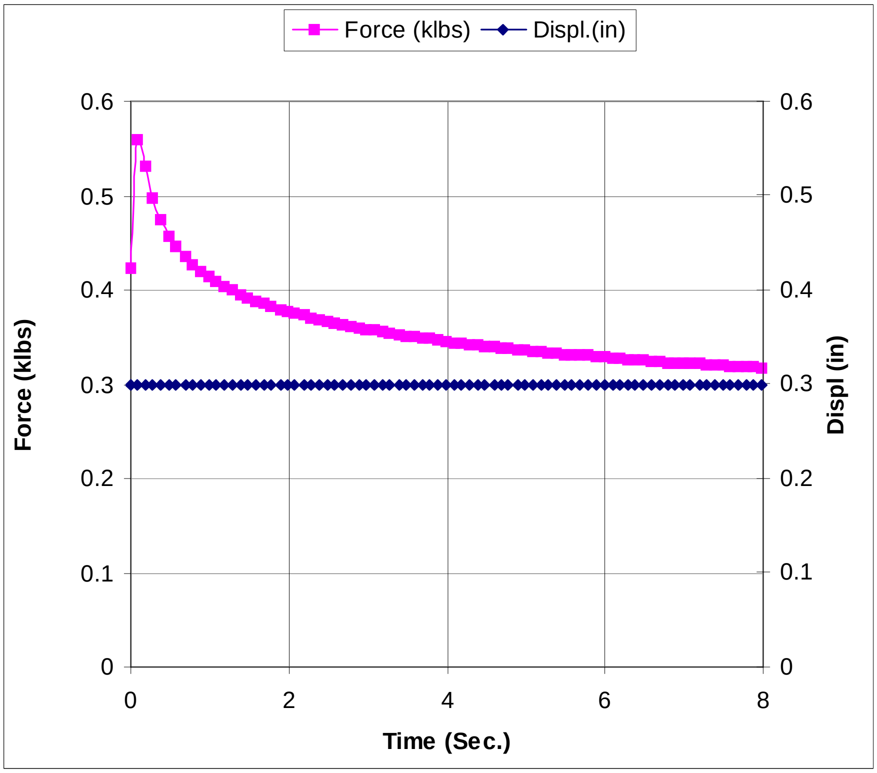

Figure 1. Short-term experimental relaxation data

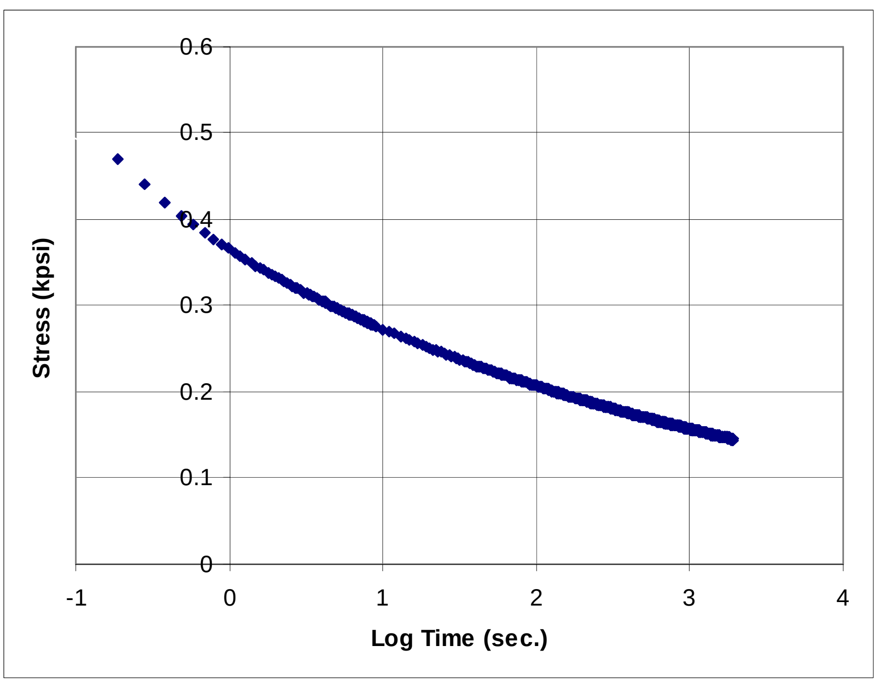

Figure 2. Long-term experimental relaxation data (semi-log plot)

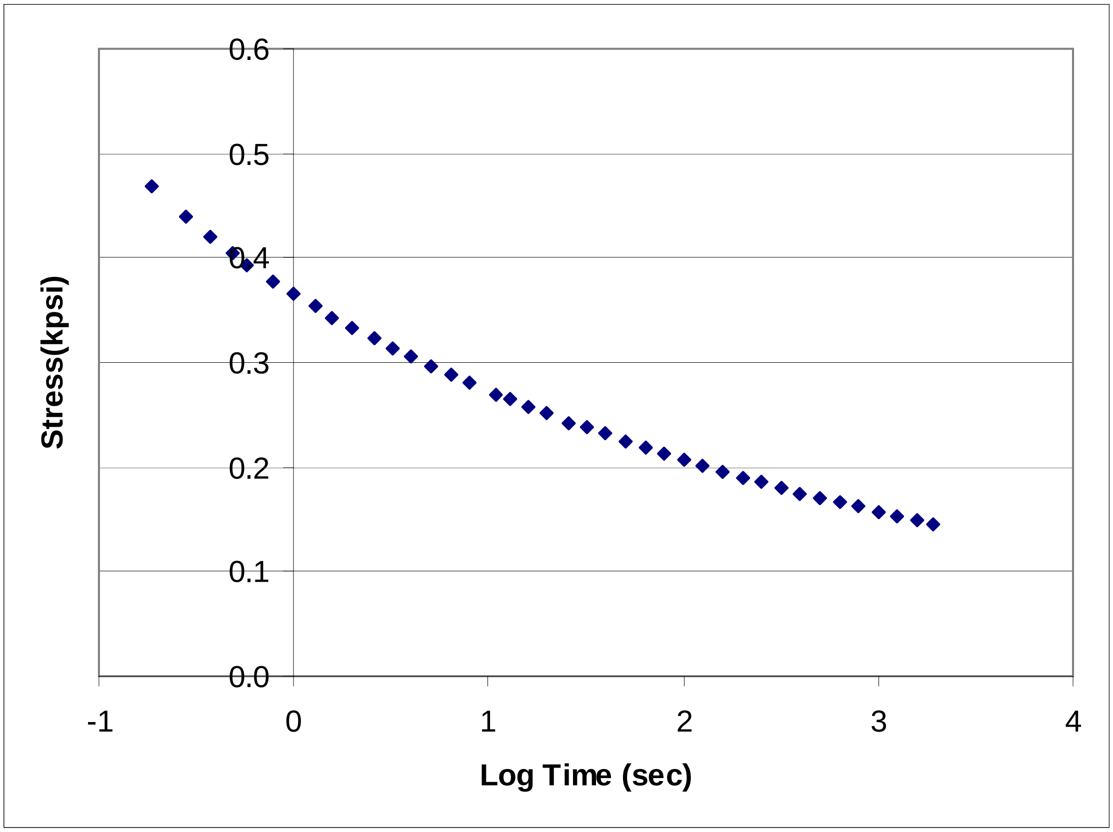

Figure 3. Selected experimental relaxation data used in the analysis

We begin the numerical study by selecting $N = 5$, five term Prony series and $\beta_1 =$ 0.00001, 0.0001, 0.001, 0.01 and 0.1. The corresponding material constants G_0, G_1, G_2,…, G_5 are determined from the linear least-square error minimization, and are shown in Table 1. Next we choose $N = 6$ and $\beta_1 =$ 0.00001, 0.0001, 0.001, 0.01 and 0.1. The corresponding material constants G_0, G_1, G_2, … , G_6 are shown in Table 2. Next we choose $N = 7$ and $\beta_1 =$ 0.00001, 0.0001, 0.001, and 0.01. The corresponding material constants G_0, G_1, G_2, … , G_7 are shown in Table 3. From these three tables, only three sets resulting from least-square error minimization are acceptable. The rest, with results show in red, are not acceptable. The best-fit coefficients of the Prony series are for $N = 5$ and $\beta_1 = 0.001$. The best-fit curve and the test data are shown in Figure 4. The test data are for a strain of 0.1; thus, the coefficients have to be multiplied by 10 for the relaxation function.

BETA	G	G	G	G	G
0.00E+00	-3.16E+05	7.25E+01	1.37E+02	1.65E+02	1.93E+02
1.00E-05	3.59E+05				
1.00E-04	-4.38E+04	8.02E+01			
1.00E-03	1.11E+03	3.49E+01	5.47E+01		
1.00E-02	-5.49E+01	6.82E+01	6.00E+01	9.78E+01	
1.00E-01	2.02E+02	6.37E+01	7.70E+01	6.27E+01	1.76E+02
1.00E+00		1.65E+02	1.33E+02	1.31E+02	5.01E+01
1.00E+01			1.30E+02	2.49E+02	7.05E+02
1.00E+02				-5.66E+05	-6.60E+09
1.00E+03					3.62E+46

Table 1. Viscoelastic material constants study: *N* = 5

BETA	G	G	G	G	G
0.00E+00	4.82E+04	1.13E+02	1.37E+02	1.65E+02	NaN
1.00E-05	-5.46E+04				
1.00E-04	6.67E+03	2.93E+01			
1.00E-03	-1.04E+02	4.84E+01	5.54E+01		
1.00E-02	7.42E+01	6.11E+01	5.85E+01	9.76E+01	
1.00E-01	6.23E+01	7.66E+01	8.07E+01	6.35E+01	NaN
1.00E+00	1.66E+02	1.33E+02	1.21E+02	1.27E+02	NaN
1.00E+01		1.30E+02	2.68E+02	3.30E+02	NaN
1.00E+02			-6.18E+05	-1.70E+09	NaN
1.00E+03				9.30E+45	NaN
1.00E+04					NaN

Table 2. Viscoelastic material constants study: *N* = 6

BETA	G	G	G	G
0.00E+00	1.85E+04	1.25E+02	1.37E+02	NaN
1.00E-05	-2.08E+04			
1.00E-04	2.54E+03	1.45E+01		
1.00E-03	-4.79E+00	5.23E+01	5.57E+01	
1.00E-02	6.34E+01	5.91E+01	5.80E+01	NaN
1.00E-01	7.61E+01	8.05E+01	8.19E+01	NaN
1.00E+00	1.33E+02	1.21E+02	1.17E+02	NaN
1.00E+01	1.30E+02	2.67E+02	3.78E+02	NaN
1.00E+02		-6.17E+05	-2.34E+09	NaN
1.00E+03			1.28E+46	NaN
1.00E+04				NaN

Table3. Viscoelastic material constants study: N = 7

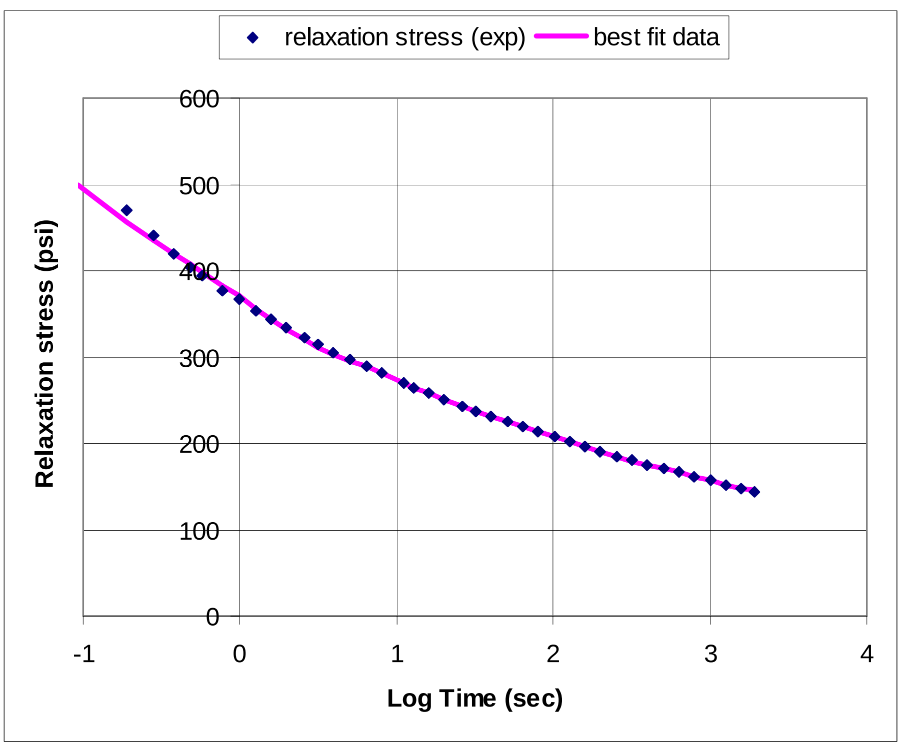

Figure 4. Comparison between constitutive equation and test data

General Guidance For Selecting β_1 And N

In general the value for β_1 is usually the inverse of the same order of the largest time in the experiment. For example, if the largest value of time in the experimental data is 10^4, then β_1 should be 0.0001. The number of terms in the Prony series should be equal to the number of decades of time covered by the experimental data. For example, if the experimental time data is from 10^{-2} to 10^4, then N should be six. It is important to check the least-square error minimization results with the test data, since there is often more than one set of results we can choose from, as shown in this paper.

Remarks

Only the simple linear viscoelastic constitutive equation is used here to demonstrate the determination of the coefficients for the Prony series. The same principle applies to other LS-DYNA® viscoelastic constitutive equations.