Comparison of Analytical and Numerical Results in Modal Analysis of Multispan Continuous Beams with LS-DYNA®

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Abstract

This paper deals with the study of natural frequencies of vibration of continuous beams supported on hinged end supports with and without overhang. The paper illustrates the analytical formulation of the natural frequencies and corresponding modes of a n-span continuous beam by using a general solution for the Euler-Bernoulli differential equation. Using two different approaches, namely the analytical method and the numerical method some typical continuous beams are analyzed and the conformance of the FEM solver LS-DYNA is tested. The objective is to test the correlation between approximated analytical and numerical methods adopted for this particular study. The computed results are given in tabular form.

Introduction

In dynamic analysis of structures the computation of natural frequencies and mode shapes is important, mostly, in the design of structures subjected to vibratory loading. In design of multispan continuous beams, it is essential that accurate determination of the natural frequencies and mode shapes are done since the structures subjected to dynamic loads is dependent on both the lower and the higher modes. In this paper, based on the analytical models, the natural frequencies and associated mode shapes of the vibrating system are obtained directly from the differential equation of motion for the undamped free transverse vibration of the continuous beam with assumptions that each span of the continuous beam is a uniform Euler-Bernoulli beam. In LS-DYNA each of the cases is solved i.e. the continuous beam having (i) beam element (ii) shell element and (iii) solid elements and frequencies obtained are compared with the analytical. The agreement between the two approaches i.e. analytical and beam LS-DYNA (using beam element) has been found to be excellent.

Multispan Continuous Beam on Simply Supports

A *n*-span continuous beam [1] has been analytical solved for the two cases preferably,

(i) $n+1$ simply supported ends without overhang
(ii) $n-1$ simply supported ends with overhang

as shown in the Fig.1(a) & 1(b). The beam is assumed to be an Euler-Bernoulli beam with constant stiffness (EI) and uniform mass distribution (m). Each span of a continuous beam is treated as an individual. The partial differential equation of motion for the undamped free transverse vibration of a uniform Euler-Bernoulli beam when the effects of tranverse shear deformation and rotary inertia are neglected is as follows:

$$EI\frac{\partial^4 y}{\partial x^4}(x,t)+m\frac{\partial^2 y}{\partial t^2}(x,t)=0 \tag{1}$$

where

x is the axial position of a point on the beam,
y(x,t) is the transverse displacement response of the beam at position *x* and time *t*,
m is the mass per unit length of the beam,
E is the elastic modulus of the material,
I is the moment of inertia of the cross-section.

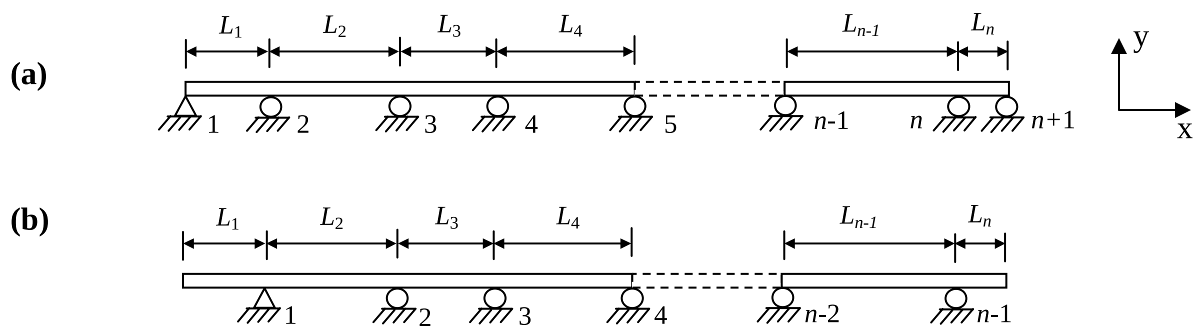


Fig. 1. n-span continuous beam layout (a) without overhang (b) with overhang

When the system performs harmonic free transverse vibration, one has

$$y(x,t)=Y(x)e^{i\omega t} \tag{2}$$

where

$Y(x)$ is the amplitude of *y(x,t)*,
ω is the angular frequency of the whole system and
$i=\sqrt{-1}$.

Substituting Eq.(2) in Eq. (1) yields the well known fourth order differential equation,

$$Y'''' - \lambda^4 Y = 0 \tag{3}$$

where

$$\lambda^4 = \frac{m\omega^2}{EI} \tag{4}$$

The general solution of the fourth order differential Eq.(3) for transverse vibration of beams may be written in the following form,

$$Y(x)=A\cosh\lambda x + B\sinh\lambda x + C\cos\lambda x + D\sin\lambda x \tag{5}$$

Eq.(5) is the displacement function for each beam segment between any two adjacent stations of a multispan beam. It is also known as the corresponding mode shape. The constant *A*, *B*, *C* and *D* can be evaluated from the boundary conditions of the spans of the continuous beam.

Determination of natural frequencies and modes shapes

For an intermediate span L_j as shown in Fig. 2, Eq. (5) can be rewritten as

$$Y_j = A_j\cosh\lambda_j x + B_j\sinh\lambda_j x + C_j\cos\lambda_j x + D_j\sin\lambda_j x \tag{6}$$

It should be noted that λ 's are given by Eq. (4) and that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_j = \cdots = \lambda_n = \lambda \tag{7}$$

In a similar manner, for span L_{j+1}

$$Y_{j+1} = A_{j+1}\cosh\lambda_{j+1}x + B_{j+1}\sinh\lambda_{j+1}x + C_{j+1}\cos\lambda_{j+1}x + D_{j+1}\sin\lambda_{j+1}x \tag{8}$$

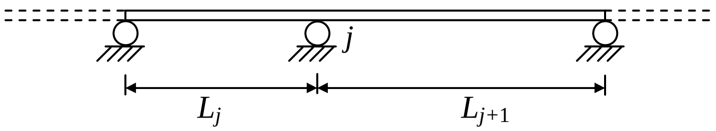


Fig.2. Two consecutive spans L_j and L_{j+1}

The first and second derivatives of Y_j (Eq. 6) w.r.t x are

$$Y_j' = A_j\lambda_j\sinh\lambda_j x + B_j\lambda_j\cosh\lambda_j x - C_j\lambda_j\sin\lambda_j x + D_j\lambda_j\cos\lambda_j x \tag{9a}$$

$$Y_j'' = A_j\lambda_j^2\cosh\lambda_j x + B_j\lambda_j^2\sinh\lambda_j x - C_j\lambda_j^2\cos\lambda_j x - D_j\lambda_j^2\sin\lambda_j x \tag{9b}$$

$$Y_j''' = A_j\lambda_j^3\sinh\lambda_j x + B_j\lambda_j^3\cosh\lambda_j x + C_j\lambda_j^3\sin\lambda_j x - D_j\lambda_j^3\cos\lambda_j x \tag{9c}$$

The first and second derivatives of Y_{j+1} (Eq. 8) w.r.t x are

$$Y_{j+1}' = A_{j+1}\lambda_{j+1}\sinh\lambda_{j+1}x + B_{j+1}\lambda_{j+1}\cosh\lambda_{j+1}x - C_{j+1}\lambda_{j+1}\sin\lambda_{j+1}x + D_{j+1}\lambda_{j+1}\cos\lambda_{j+1}x \tag{10a}$$

$$Y_{j+1}'' = A_{j+1}\lambda_{j+1}^2\cosh\lambda_{j+1}x + B_{j+1}\lambda_{j+1}^2\sinh\lambda_{j+1}x - C_{j+1}\lambda_{j+1}^2\cos\lambda_{j+1}x - D_{j+1}\lambda_{j+1}^2\sin\lambda_{j+1}x \tag{10b}$$

$$Y_{j+1}''' = A_{j+1}\lambda_{j+1}^3\sinh\lambda_{j+1}x + B_{j+1}\lambda_{j+1}^3\cosh\lambda_{j+1}x + C_{j+1}\lambda_{j+1}^3\sin\lambda_{j+1}x - D_{j+1}\lambda_{j+1}^3\cos\lambda_{j+1}x \tag{10c}$$

Without Overhang

The boundary conditions (BCs) at n+1 supports are:

(i) $$Y_j\big|_{x=0} = Y_j\big|_{x=L_j} = 0 \tag{11a,b}$$

(ii) The continuity conditions can be written for any two consecutive spans L_j & L_{j+1}. These conditions suggest that the rotations and moments of the two spans at support $j+1$ should be equal i.e.

$$Y_j'\big|_{x=L_j} = Y_{j+1}'\big|_{x=0} = 0 \tag{11c}$$

$$Y_j''\big|_{x=L_j} = Y_{j+1}''\big|_{x=0} = 0 \tag{11d}$$

Applying BCs (Eq. 11) to Eqs. (8), (9) and (10) we get the following governing equations for each pair of adjacent spans after simplifying,

Table 1

No	Span	Equation
1	1-2	$-C_2(\beta_1+\beta_2)+C_3\gamma_2=0$
2	2-3	$C_2\gamma_2-C_3(\beta_2+\beta_3)+C_4\gamma_3=0$
3	3-4	$C_3\gamma_3-C_4(\beta_3+\beta_4)+C_5\gamma_4=0$
	………………………………………………	
n-2	n-2 & n-1	$C_{n-2}\gamma_{n-2}-C_{n-1}(\beta_{n-2}+\beta_{n-1})+C_n\gamma_{n-1}=0$
n-1	n-1 & n	$-C_{n-1}\gamma_{n-1}+C_n(\beta_{n-1}+\beta_n)=0$

where $$\beta_j = \coth \lambda_j L_j - \cot \lambda_j L_j \tag{12a}$$

$$\gamma_j = \mathrm{csch} \lambda_j L_j - \csc \lambda_j L_j \tag{12b}$$

The product $\lambda L_2, \lambda L_3, \cdots \lambda L_n$ can be expressed in terms of λL_1 as per Eq. (7).

With overhang

The boundary conditions at the free ends i.e
(i) at x=0 of span '1' are

$$Y_1'' \big|_{x=0} = 0 \quad \text{(zero bending moment)} \tag{13a}$$

$$Y_1''' \big|_{x=0} = 0 \quad \text{(zero shear force)} \tag{13b}$$

(ii) at x=L_n of span 'n' are

$$Y_n'' \big|_{x=L_n} = 0 \quad \text{(zero bending moment)} \tag{13c}$$

$$Y_n''' \big|_{x=L_n} = 0 \quad \text{(zero shear force)} \tag{13d}$$

The boundary conditions at n-1 supports

$$Y_j \big|_{x=L_j} = Y_{j+1} \big|_{x=0} = 0 \quad \text{, j=1, 2, 3,......, n-1} \tag{14a,b}$$

$$Y_j' \big|_{x=L_j} = Y_{j+1}' \big|_{x=0} \quad \text{, j=1, 2, 3,......, n-1} \tag{14c}$$

$$Y_j'' \big|_{x=L_j} = Y_{j+1}'' \big|_{x=0} \quad \text{, j=1, 2, 3,......, n-1} \tag{14d}$$

Applying BCs (Eqs. 13 &14) to Eqs. (8), (9) and (10) we get the following governing equations as shown in Table 2, for each pair of adjacent spans after simplifying,

Table 2

No	Span	Equation
1	1-2	$-C_2(2\alpha_1 + \beta_2) + C_3\gamma_2 = 0$
2	2-3	$C_2\gamma_2 - C_3(\beta_2 + \beta_3) + C_4\gamma_3 = 0$
3	3-4	$C_3\gamma_3 - C_4(\beta_3 + \beta_4) + C_5\gamma_4 = 0$
	....	
n-2	n-2 & n-1	$C_{n-2}\gamma_{n-2} - C_{n-1}(\beta_{n-2} + \beta_{n-1}) + C_n\gamma_{n-1} = 0$
n-1	n-1 & n	$-C_{n-1}\gamma_{n-1} + C_n(\beta_{n-1} + 2\alpha_n) = 0$

where $$\alpha_1 = \frac{1 + \cosh \lambda_1 L_1 \cos \lambda_1 L_1}{\sinh \lambda_1 L_1 \cos \lambda_1 L_1 - \cosh \lambda_1 L_1 \sin \lambda_1 L_1} \tag{15a}$$

$$\alpha_n = \frac{1 + \cosh \lambda_n L_n \cos \lambda_n L_n}{\sinh \lambda_n L_n \cos \lambda_n L_n - \cosh \lambda_n L_n \sin \lambda_n L_n} \tag{15b}$$

Non Trivial Solution

For a non trivial solution, the determinant of the coefficients of $C_2, C_3, \cdots C_n$ in the Table 1 must be equal to zero i.e.,

$$\begin{vmatrix} -(\beta_1+\beta_2) & \gamma_2 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ \gamma_2 & -(\beta_2+\beta_3) & \gamma_3 & 0 & \cdots\cdots & 0 & 0 & 0 \\ & \gamma_3 & -(\beta_3+\beta_4) & \gamma_4 & \cdots\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & \gamma_{n-2} & -(\beta_{n-2}+\beta_{n-1}) & \gamma_{n-1} \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & -\gamma_{n-1} & \beta_{n-1}+\beta_n \end{vmatrix} = 0 \tag{16}$$

Similarly, in case of Table 2,

$$\begin{vmatrix} -(2\alpha_1+\beta_2) & \gamma_2 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ \gamma_2 & -(\beta_2+\beta_3) & \gamma_3 & 0 & \cdots\cdots & 0 & 0 & 0 \\ & \gamma_3 & -(\beta_3+\beta_4) & \gamma_4 & \cdots\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & \gamma_{n-2} & -(\beta_{n-2}+\beta_{n-1}) & \gamma_{n-1} \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & -\gamma_{n-1} & \beta_{n-1}+2\alpha_n \end{vmatrix} = 0 \tag{17}$$

The determinants so obtained i.e. Eq. 16 & Eq. 17 are called the *frequency determinants* which on expansion yields the frequency equation,

$$F(\lambda) = 0 \tag{18}$$

No simple expression for the roots of Eq. (18) is available. A simple way to determine the values of λL_1 that satisfy the frequency equation is to graphically plot λL_1 versus $F(\lambda)$ for various values of λL_1. The roots of the equation are $p_1, p_2, p_3 \cdots\cdots$ such that $\lambda L_1 = p_i,\quad i \in I$ .The corresponding natural frequencies (f_i) and eigenvalues (e_i) of the beam are respectively given by the expressions,

$$f_i = \frac{\omega_i}{2\pi} \tag{19}$$

and

$$e_i = \omega_i^2 \tag{20}$$

where

$$\omega_i = \lambda^2 \sqrt{\frac{EI}{m}} \tag{21}$$

Illustrative Examples

Two cases of the continuous beam configuration have been illustrated here. The beam has a uniform rectangular cross-sectional area over a length of 7.5 meters and is hinged at unequal intervals as shown in Fig. 3 & 4.

a) 6 unequal span continuous beam without overhang as shown in Fig 3.

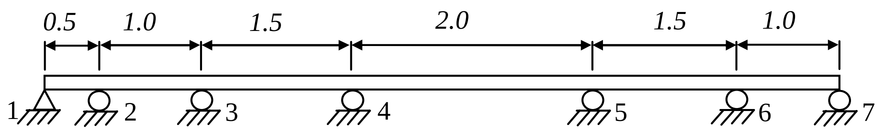


Fig. 3. 6-span continuous single beam without overhang

b) 6 unequal span continuous beam with overhang as shown in Fig 4.

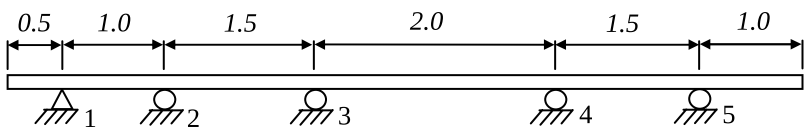


Fig. 4. 6-span continuous single beam with overhang

Table 3

Parameters	Value
Elastic modulus of the material, E (GPa)	210
Density of the material, ρ (Kg/m^3)	7860
Area of cross section, A (m^2)	0.05 (width) x 0.08 (depth)
Length of continuous beam ,L (m)	7.5
Moment of inertia of the cross-section, I (m^4)	21.33 E-7
Mass per unit length, m (Kg/m)	31.44
Poisson's ratio	0.3

The above examples are solved analytically as well as using FEM code LS-DYNA. The analytical formulation has been solved in the commercial code MATLAB [2] to obtain a graphical plot of λL_1 vs $F(\lambda)$ and subsequently the frequencies. In LS- DYNA, different types of elements have been used to solve the problem i.e. the continuous beam having (i) beam elements (ii) shell elements namely quads and (iii) solid brick elements. The continuous beam is made of stainless steel. Table 3 shows some of the important parameters required to illustrate the example. The material property of the beam has been implemented in LS-DYNA [3], using MAT #001 (*MAT_ELASTIC). The nodes at the hinged points are constrained i.e. translational constraint in x, y & z direction and rotational constraint in x & y direction. The computer time is relatively short for beam and shell element model since it has fewer degrees of freedom whereas for solid elements the computation time is relatively more due to more number of elements and nodes in the model. Each 1 second simulation required about 5-120 seconds of the CPU on a SGI ONYX4 IRIX Workstation. The values of natural frequencies (transverse direction) obtained by analytical method are compared with the FE Model analyzed in LS-DYNA as shown in Table 4. Fig. 5 & 6 shows the mode shapes in transverse direction of the 6-span continuous beam with and without overhang.

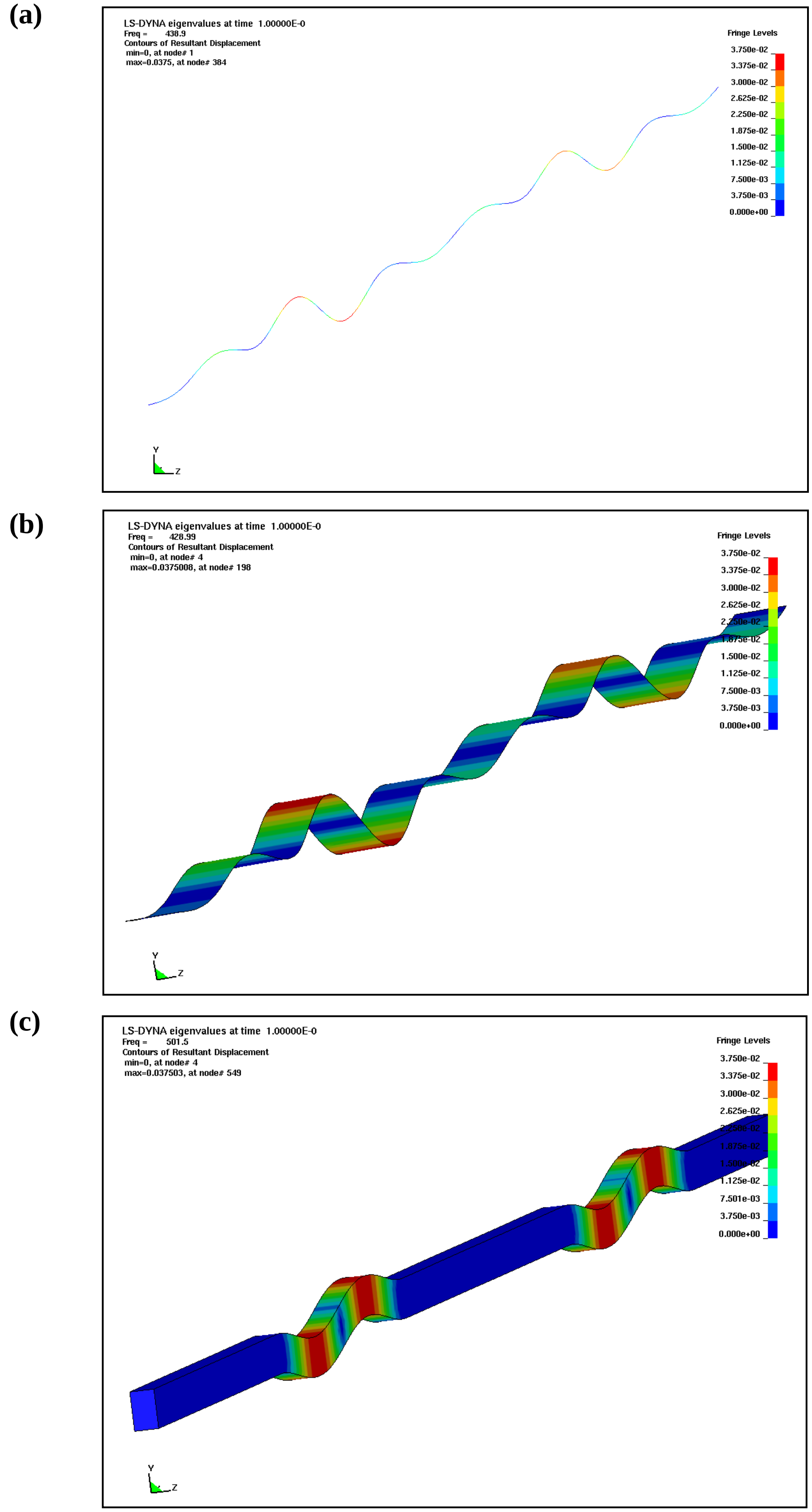
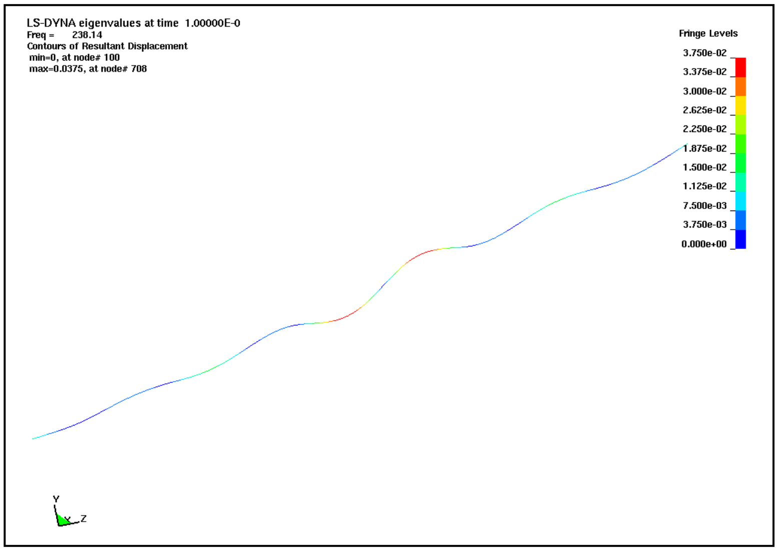
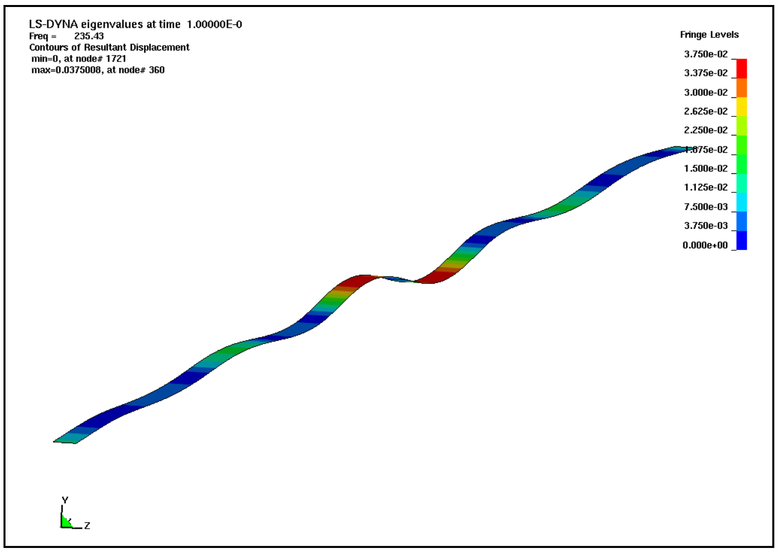


Fig. 5. Eighth mode shape in transverse direction of 6-span beam without overhang (a) Beam element (b) Shell element and (c) Solid brick element (Scale Factor for Eigen Vector Animation is 10)

(a)



(b)



(c)

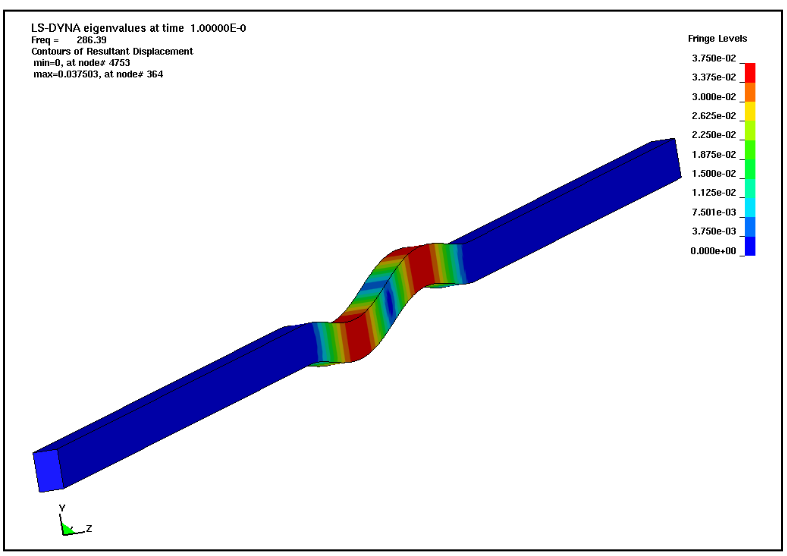


Fig. 6. Sixth mode shape in transverse direction of 6-span beam with overhang (a) Beam element (b) Shell element and (c) Solid brick element (Scale Factor for Eigen Vector Animation is 10)

Table 4. Comparison of Frequencies (transverse direction) of multi-span continuous beam (Two cases mentioned above)

	Frequencies (Hz)			
	Analytical	LS-DYNA®		
		Beam Element ELFORM=4 (Belystschko -Schwer full cross-section integration) Nodes :1501 Elements:750 Nodal mass = 2.3391E+02 Kg (without overhang) Nodal mass = 2.3423E+02 Kg (with overhang)	**Shell Element** ELFORM=2 (Belystschko Tsay) Nodes : 4506 Elements: 3750 Nodal mass = 2.3391E+02 Kg (without overhang) Nodal mass = 2.3423E+02 Kg (with overhang)	**Solid Element** ELFORM=3 (Fully integrated quadratic 8 node solid element with nodal rotations.) Nodes : 40554 Elements: 30000 Nodal mass = 2.3391E+02 Kg (without overhang) Nodal mass = 2.3423E+02 Kg (with overhang)
Without Overhang	66.8484	66.841	66.606	105.577
	119.349	119.348	118.643	186.281
	137.2702	137.267	136.141	186.282
	226.0765	226.267	223.813	287.070
	251.8323	251.826	248.429	409.774
	289.7076	289.710	285.014	410.283
	402.3566	402.360	395.195	501.490
	438.9103	438.901	428.986	501.495
	525.921	525.899	512.182	552.861
	781.1456	781.121	758.085	894.367
With Overhang	38.2006	38.205	38.159	66.583
	68.1519	68.149	67.905	105.327
	114.7094	114.702	114.211	185.840
	135.0932	135.091	134.193	185.840
	151.8342	151.832	150.513	262.890
	238.1615	238.143	235.425	286.388
	316.2866	316.292	311.303	405.415
	330.8691	330.860	326.473	409.311
	416.0047	415.984	407.944	500.300
	452.1027	452.073	441.346	500.305

Conclusion

Table 4 shows that the frequencies obtained by the FEM solver LS-DYNA, using solid brick elements somewhat deviates from that of the analytical (exact method). Although quadratic 8 node solid elements have no more integration points than a linear element, it yields better results at less expense than the linear elements. But it gives less accurate results compare to beam and shell elements due to variations in stiffness while using different formulations (frequencies depend on stiffness and mass). The degrees of freedom are different for beam, shell and solid elements. The bending deformation of the model having quadratic 8 node solid elements is stiff and not more accurate than shell or beam elements. So the frequencies of the model with solid elements are higher. On the other hand, the agreement between the two approaches i.e. analytical and beam LS-DYNA (using beam element) has been found to be excellent. In general, the natural frequencies of the continuous beams can be calculated more accurately by the use of beam elements rather than using solid brick elements in LS-DYNA. But it is not always possible to model a complex geometry using beam elements. The results also showed the reliability and capability of the dynamic code LS-DYNA which can be further utilized to solve problems of bigger and complex nature. This study provided a basis for studying the modal analysis of 4-rod type Radio Frequency Quadruple (RFQ) structure implementing boundary conditions as in actual scenario. But the above topic is beyond the scope of discussion in this paper.

References

[1] Fertis Demeter G., Dynamics and Vibration of Structures, John Wiley & Sons, Inc.,1973.
[2] Using Matlab (Version 7), The Mathwork, Inc., Natick MA 01760.
[3] LS-DYNA® Keyword User's Manual (Version 970), Livermore Software Technology Corporation, Livermore, CA 94551, April 2003.