Preliminary Results for an Isogeometric Shell

David J. Benson

Dept. of Structures, UCSD

Yuri Bazilevs

Thomas J. R. Hughes

ICES, U Texas Austin

Research supported by NSF grant 0700204

Additional support for Bazilevs and Hughes from ONR

Isogeometric Analysis

- Isogeometric analysis uses NURBS as basis functions.
- NURBS are the basis functions used in CAD programs.
- Therefore: facilitates direct CAD to analysis interface.
- NURBS are nicely behaved.
  - Improved numerical conditioning.
  - Larger time step size for higher order elements than for Lagrangian polynmials.

B-Spline Basis Functions

- Piecewise polynomials in space.
- Degree determined the the knot vector:

$$\Xi = \{\xi_1, \ldots, \xi_{n+p+1}\}$$

- Coefficients of polynomials are points in space, referred to as *control points*, B_i
- Basis functions are generated recursively using the knot vector starting at *p*=0 (piecewise constants).

$$N_{i,0}(\xi) = \begin{cases} 1 \text{ if } \xi_i \leq \xi < \xi_{i+1} \\ 0 \text{ otherwise.} \end{cases}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi).$$

Properties of B-Splines

- B-splines sum to 1 like Lagrange interpolation functions.

$$\sum_{i=1}^{n} N_{i,p}(\xi) = 1 \ \forall \xi$$

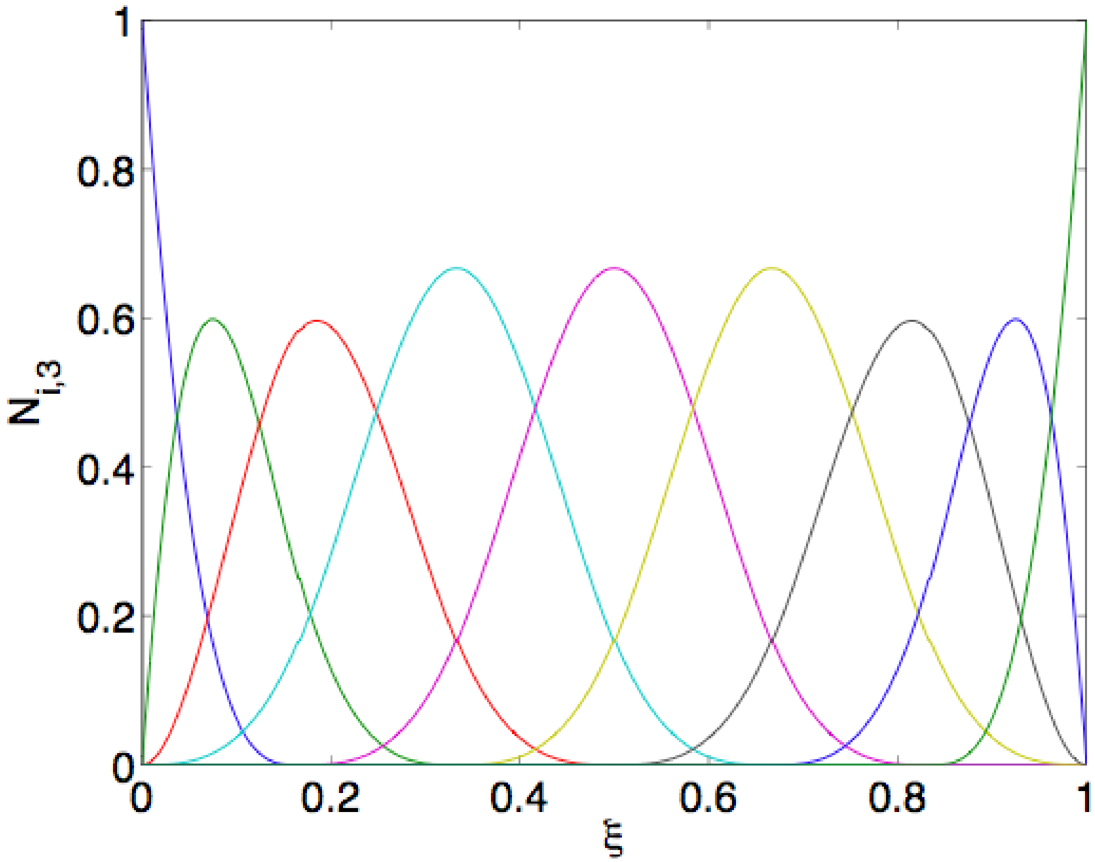
- The support of each $N_{i,p}(\xi)$ compact and contained in the interval $[\xi_i, \xi_{i+p+1}]$ similar to Lagrange interpolation polynomials.
- B-spline basis functions are non-negative:

$$N_{i,p}(\xi) \geq 0 \ \forall \xi$$

(in contrast to higher order Lagrange polynomials).

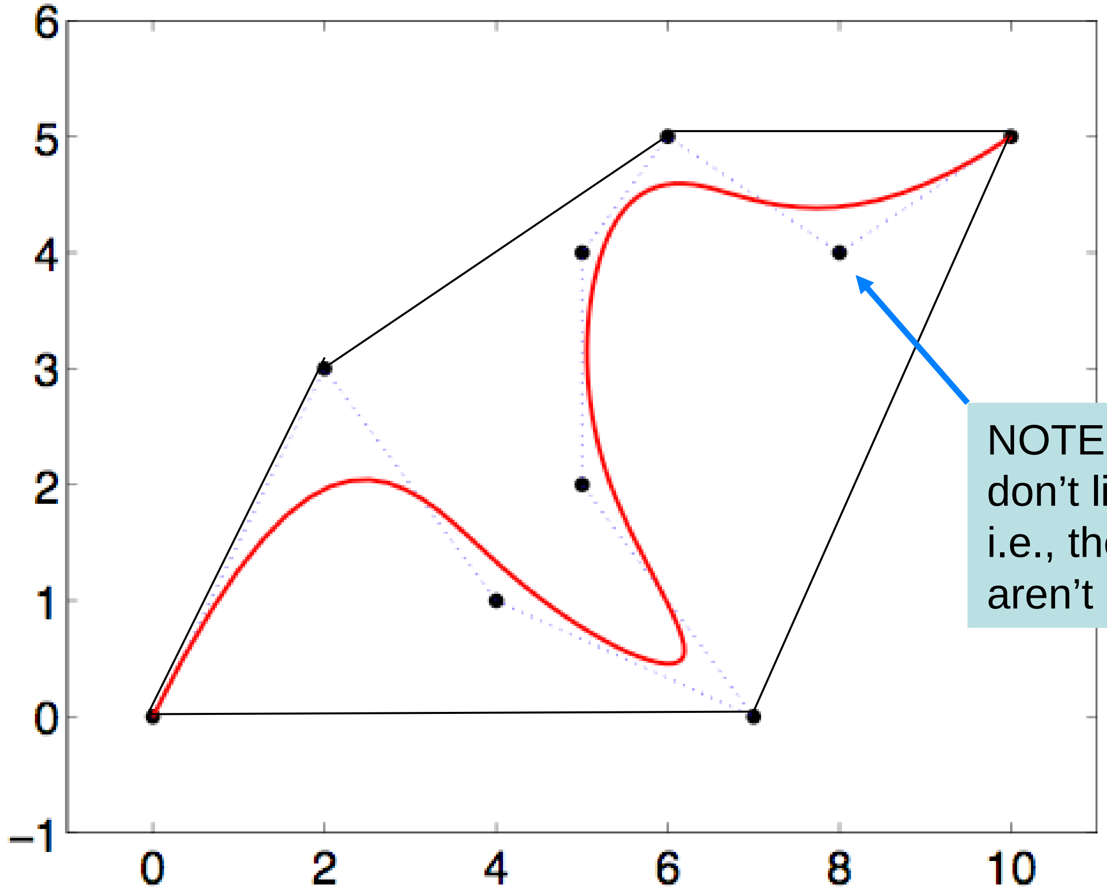
Cubic B-Spline Basis Functions

$$\Xi = \{0, 0, 0, 0, 1/6, 1/3, 1/2, 2/3, 5/6, 1, 1, 1, 1\}$$



Positivity Limits Oscillations

Curve lies inside the convex hull of the control points.



NOTE: Control points don't lie on the curve, i.e., the basis functions aren't interpolatory.

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{B}_i$$

B-Spline Surfaces and Solids

- Surfaces and solids are described in terms of tensor products of one-dimensional basis functions as is standard with Lagrange interpolation functions in standard FEA.

$$\mathbf{S}(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)\mathbf{B}_{i,j}$$ Surface

$$\mathbf{S}(\xi,\eta,\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} N_{i,p}(\xi)M_{j,q}(\eta)L_{k,l}(\zeta)\mathbf{B}_{i,j,k}$$ Solid

NURBS

- Non-Uniform Rational B-Splines (NURBS)
- Control points are homogenous coordinates.

$$(\mathbf{B}_i)_j = \frac{(\mathbf{B}_i^w)_j}{w_i},\ j = 1, \ldots, d \qquad w_i \text{ =weights}$$

- Basis functions: $R_i^p(\xi) = \dfrac{N_{i,p}(\xi) w_i}{\sum_{\hat{i}=1}^{n} N_{\hat{i},p}(\xi) w_{\hat{i}}}$

- Curve: $\mathbf{C}(\xi) = \sum_{i=1}^{n} R_i^p(\xi) \mathbf{B}_i$

NURBS Properties

- NURBS basis functions sum to 1.

$$\sum_{i=1}^{n} R_i^p(\xi) = 1 \; \forall \xi$$

- Continuity and support are the same as B-Splines.
- If all weights are equal, NURBS become B-Splines.
- NURBS surfaces and solids are tensor products.
- Basis functions are specified in terms of a knot vector like B-Splines.

Isogeometric Analysis

- A mesh for a NURBS patch is defined by the product of knot vectors.
- Knot spans subdivide the domain into “elements.”
- The support of each basis function consists of a small number of elements.
- The control points associated with the basis functions define the geometry.
- The isoparametric concept is invoked, that is, the unknown variables are represented in terms of the basis functions which define the geometry.

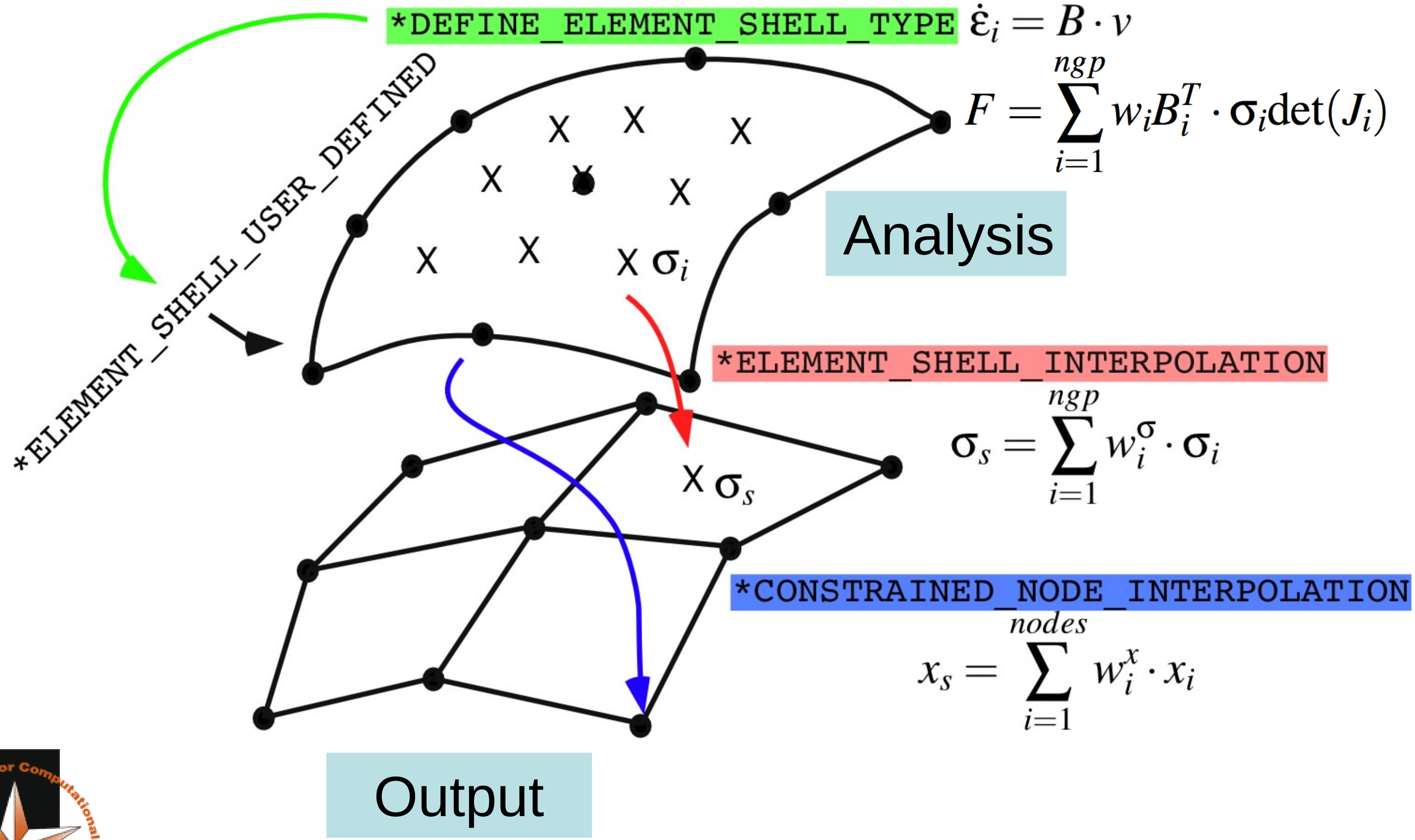
Shell Formulations

- 3 types currently available.
- IFORM=0: Degenerated solid element with rotational DOF.
  - $$v_i(\xi) = \sum_{A=1}^{n} N_A(\xi)\left(v_{Ai} + \frac{h\xi_3}{2} e_{ijk}\omega_{Aj} n_{Ak}\right)$$
- IFORM=2: Thin shell <u>without</u> rotational DOF.
  - $$v_i(\xi) = \sum_{A=1}^{n} N_A(\xi) v_{Ai} + \frac{h\xi_3}{2} \sum_{B,k} \frac{\partial n_i(\xi)}{\partial x_{Bk}} v_{Bk}$$
- IFORM=3: Reissner-Mindlin with rotational DOF.
  - $$v_i(\xi) = \sum_{A=1}^{n} N_A(\xi)\left(v_{Ai} + \frac{h\xi_3}{2} e_{ijk}\omega_{Aj} n_k(\xi)\right)$$

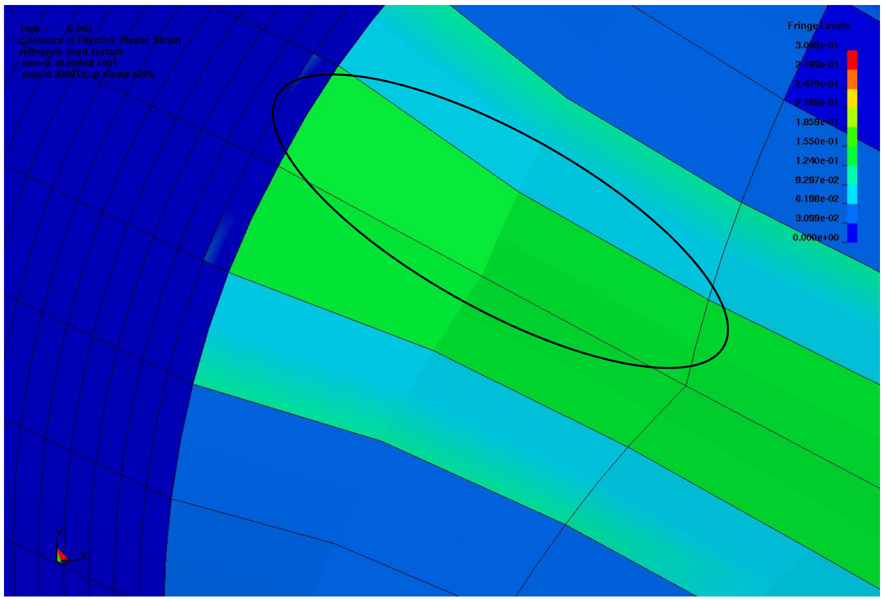
Implementation in LS-DYNA

- User-defined element capability:
  - Fast prototyping of elements without programming.
  - Available for both solids and shells.
  - Executes slower than production elements.
- Some boundary conditions implemented via interpolation elements.
  - Contact doesn't have underlying smoothness of NURBS.
  - Pressure distribution is not exactly integrated.

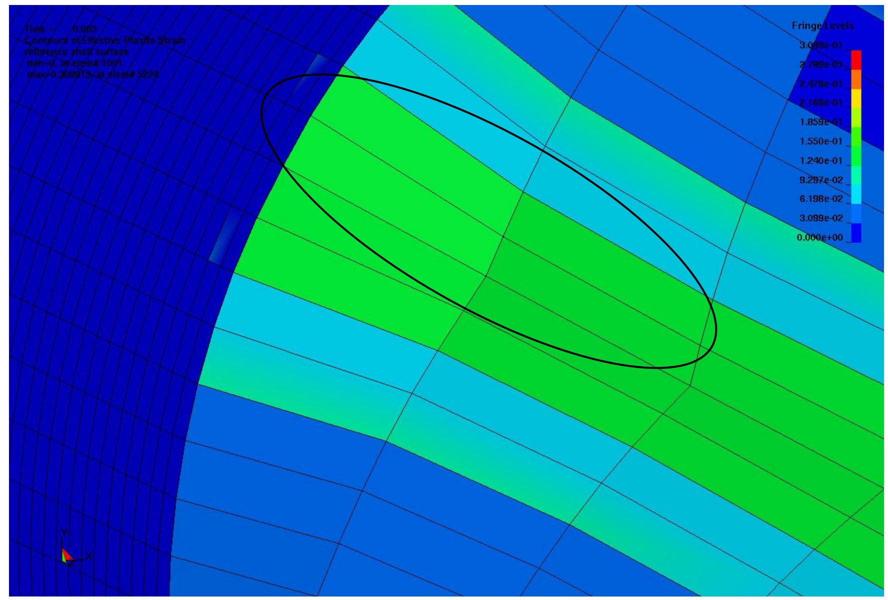
User-Defined Elements



User-defined Elements



User-defined element



Four interpolation elements

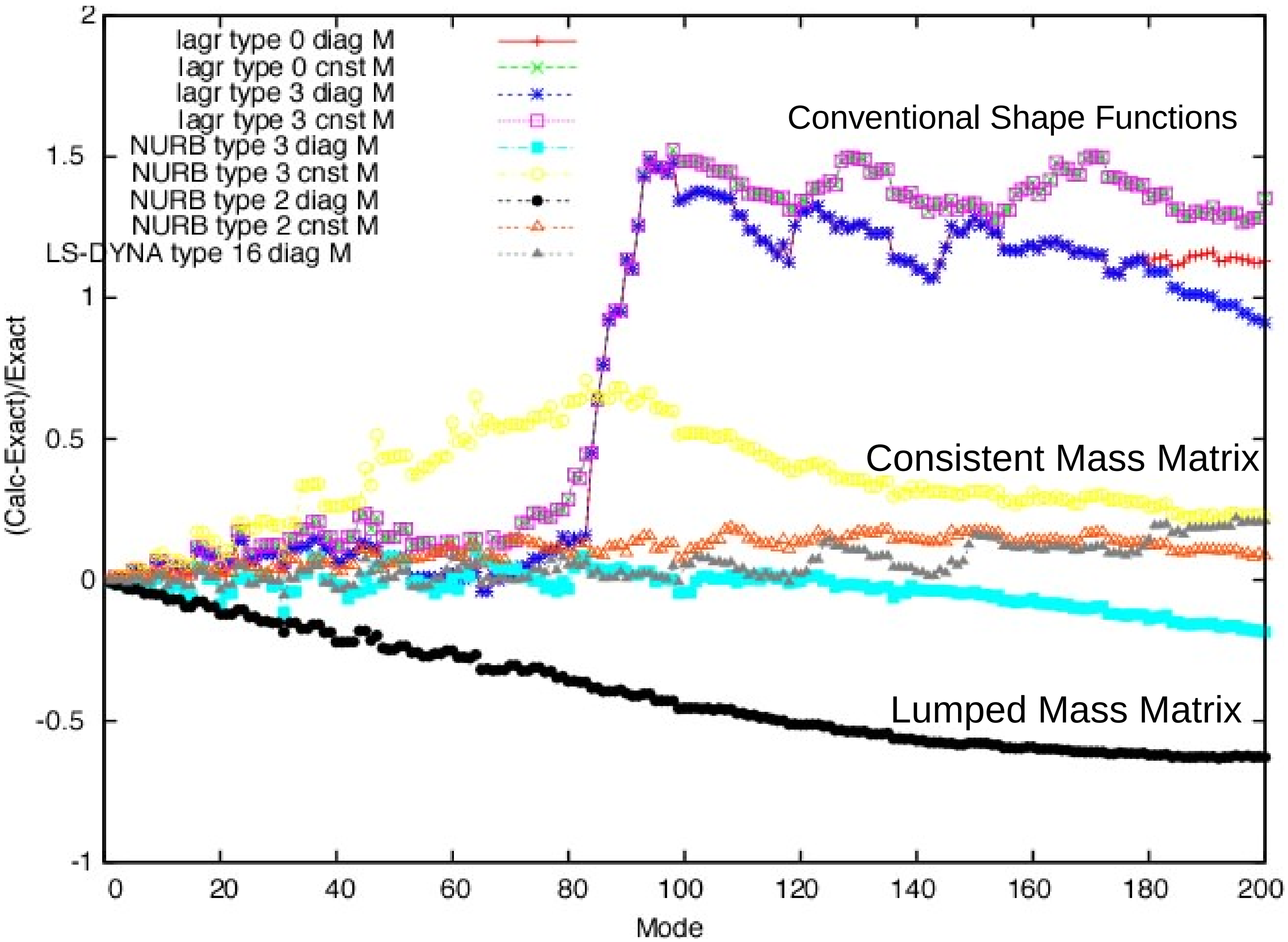
Later results display interpolation elements to better show the deformed geometry.

Transverse Vibrations of a Poisson-Kirchhoff Plate

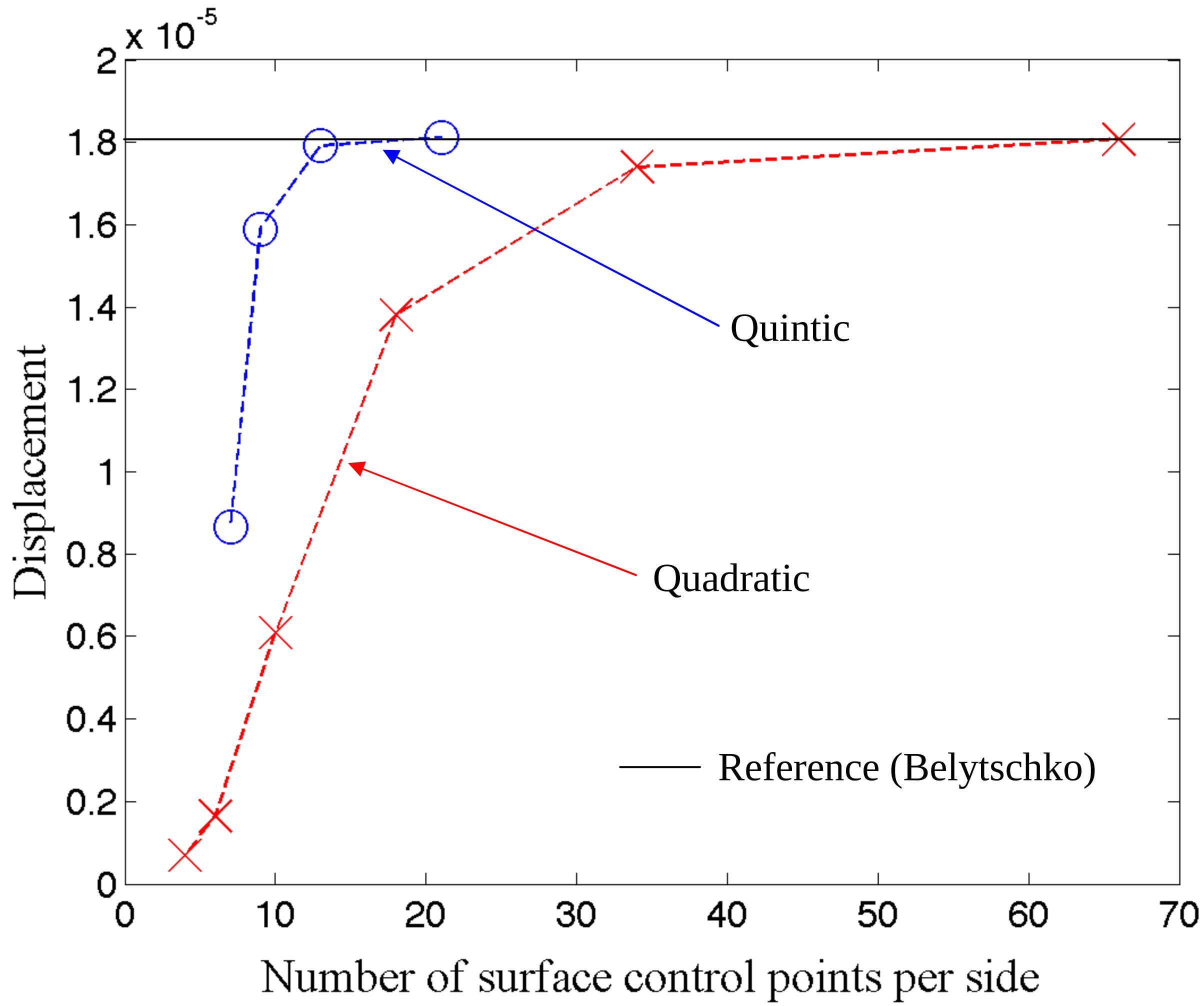
- Square plate, simply supported.
  - [0,1]x[0,1]
  - Assumes unit flexural stiffness & density.
  - Exact solution: $\omega_{mn} = \pi^2(m^2 + n^2)$
- J. A. Cottrell et al., *CMAME* 195 (2006) 5257-5296:
  - Reference for analytical solution.
  - Solution with isogeometric solids.

Eigenvalue Results:

Simply Supported Flat Plate with Quadratic NURBS



Pinched Cylinder Displacement Convergence for NURBS Meshes



Flat Plate Problem

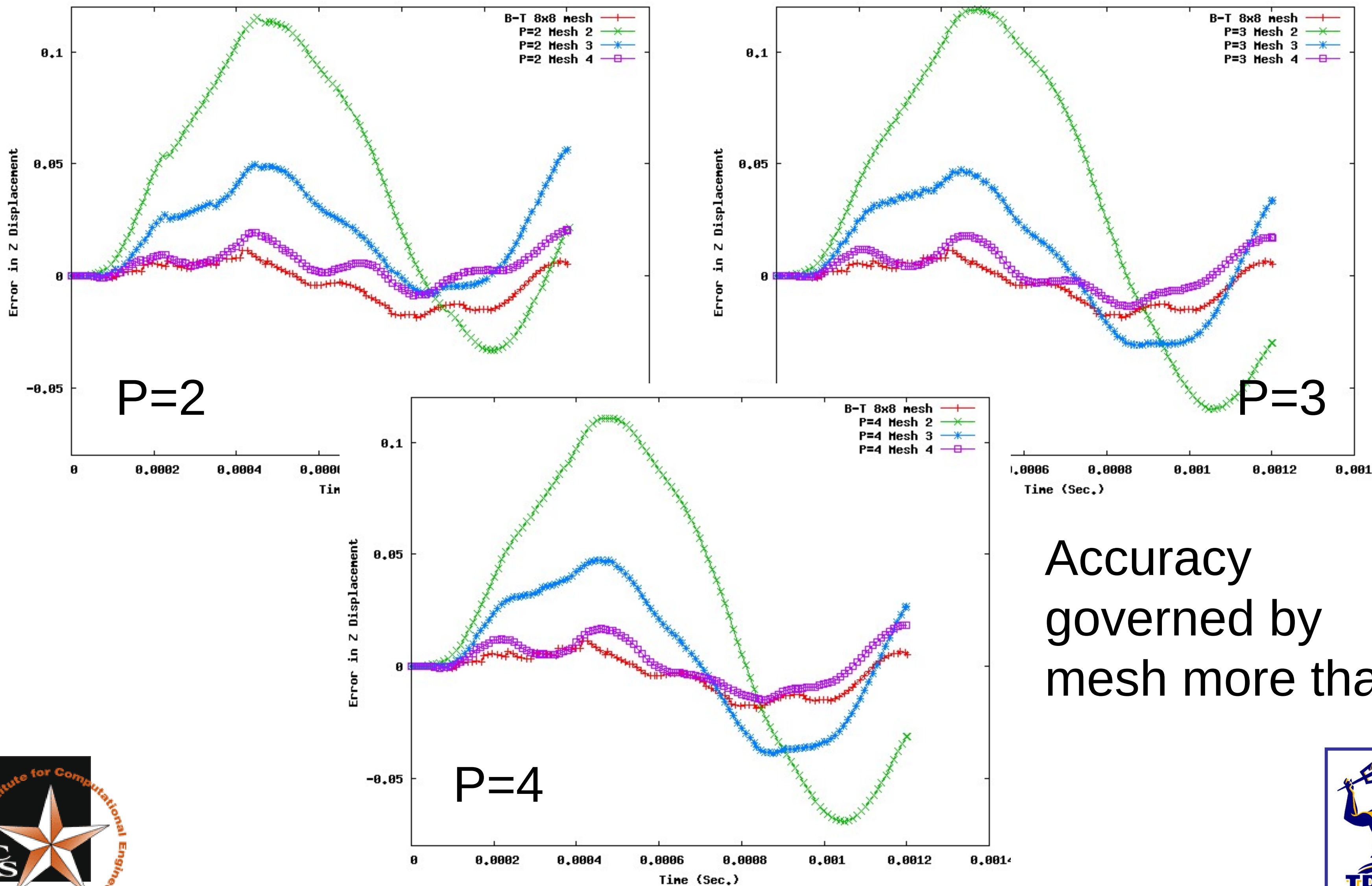
- Data from Belytschko et al., *CMAME* 42 (1984) 225-251.
- Simply supported 10 x 10 plate subjected to uniform pressure load of 300 psi.
- Reference solution: 64 x 64 B-T elements.
- Isogeometric solutions for P=2,3,4
- IFORM=3

Flat Plate Problem
Isogeometric Meshes

	P=2	P=3	P=4
M=2x2	16 CP	25 CP	36 CP
M=4x4	36 CP	49 CP	64 CP
M=8x8	100 CP	121 CP	144 CP

In 1-D: # Control Points (CP) = P + # Elements

Flat Plate Convergence Study

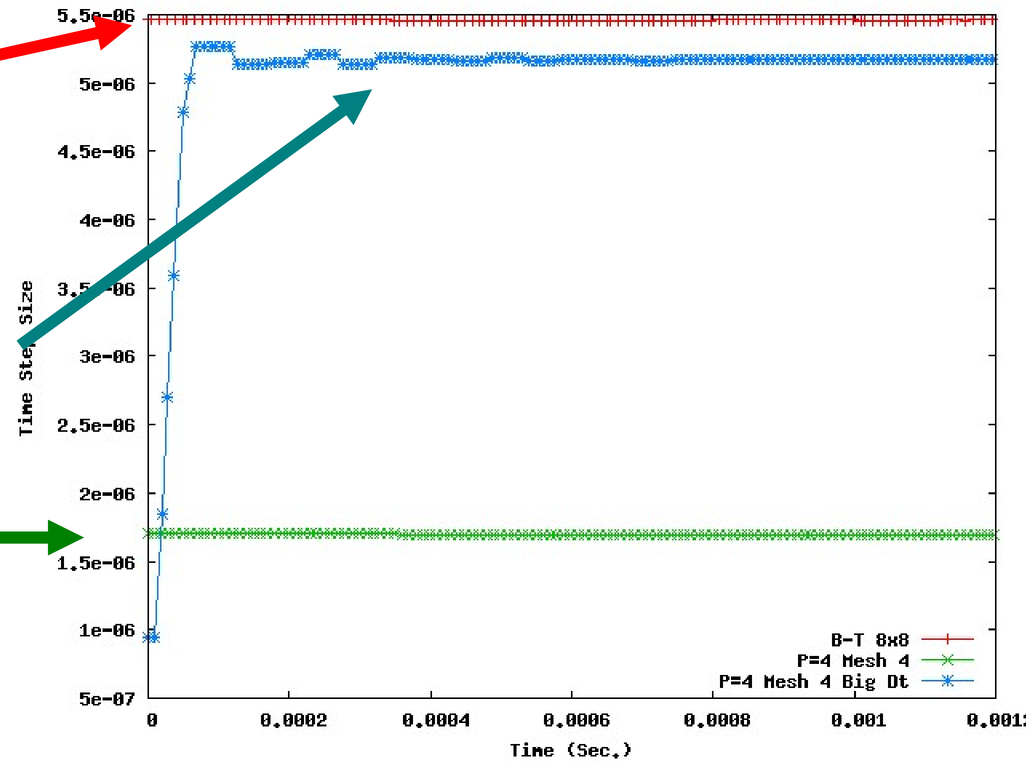


Accuracy governed by mesh more than P

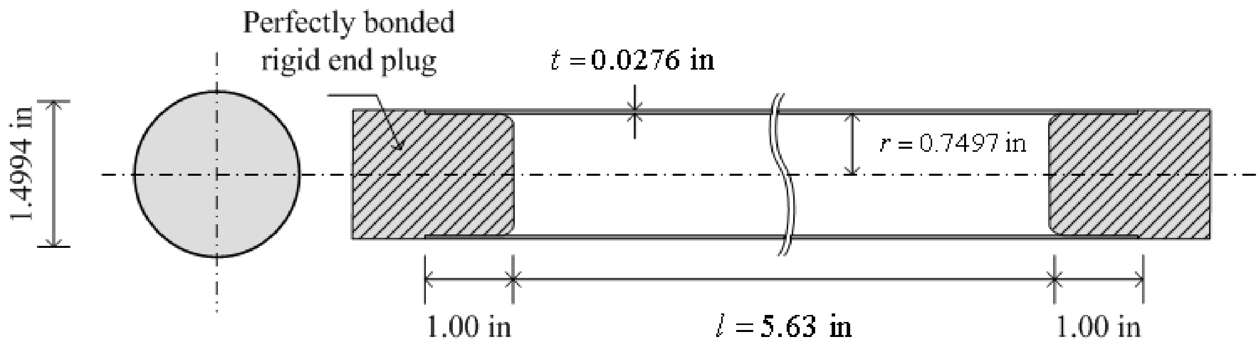
Flat Plate Time Step Size

Preliminary Result: Higher Order NURBS Have Dt Almost as Large as Linear Elements

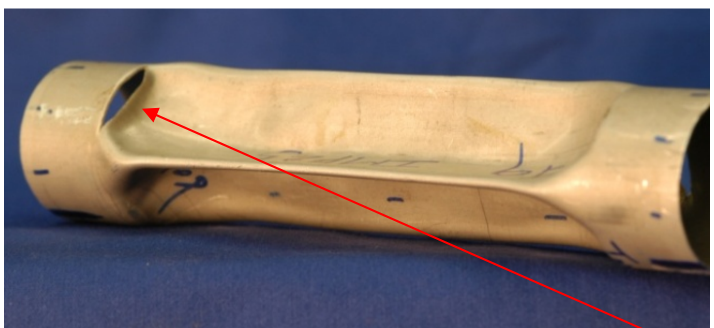
- Linear B-T element: $dt=5.5x10^{-6}$
- Quartic NURBS using maximum eigenvalue: $dt=1/\omega=5x10^{-6}$
- Quartic NURBS element based on control point spacing: $dt=1.7x10^{-6}$

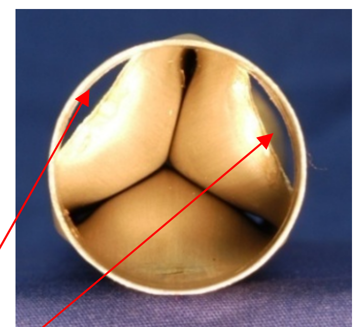


Texas Experiment(IMP25): Short Cylinder (Mode 3) (1)



- *Experiment:* $p_{cr} = 410\,\text{psi}$





(Fracture at the ends was observed.)

- *Analytic solution:* $p_{cr} = 441.43\,\text{psi}$

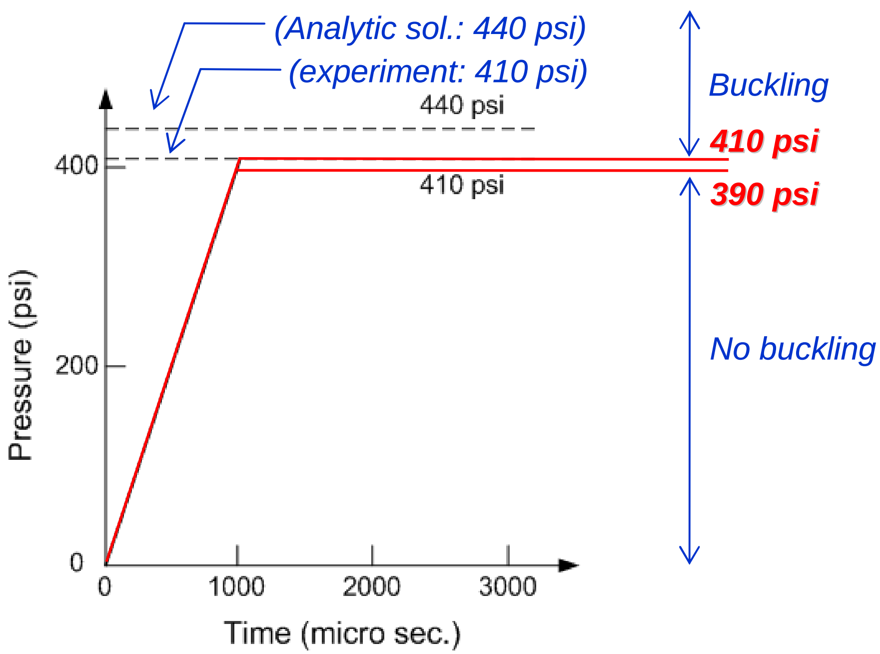
- *Computed critical pressure:* $p_{cr} = 410\,\text{psi}$

Data courtesy of Ted Belytschko
Experimental results courtesy of Stelios Kyriakides, Univ Texas

Texas Experiment(IMP25): Short Cylinder (Mode 3) (3)

- Prediction of buckling pressure:

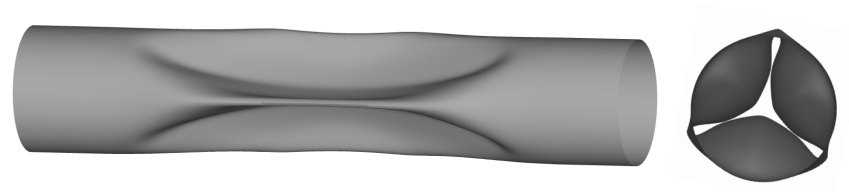
(Numerical prediction)



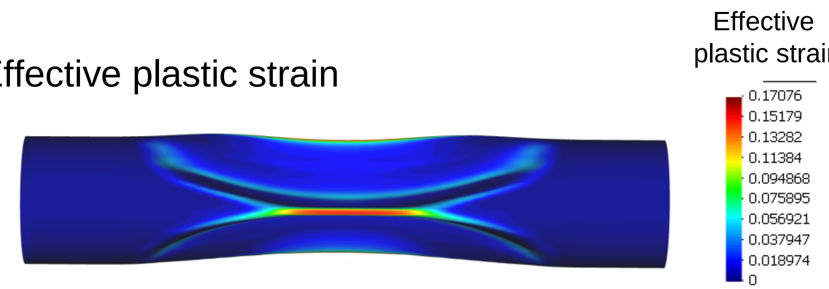
- Experiment: $p_{cr} = 410\,\text{psi}$

$p = 410\,\text{psi}$ (t= 20.0 ms)

Deformed shape

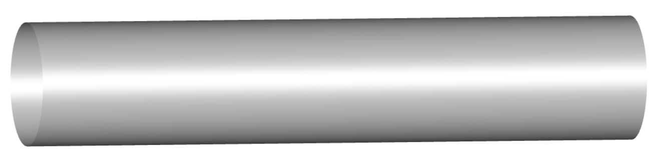


Effective plastic strain



$p = 390\,\text{psi}$ (t= 20.0ms)

Deformed shape



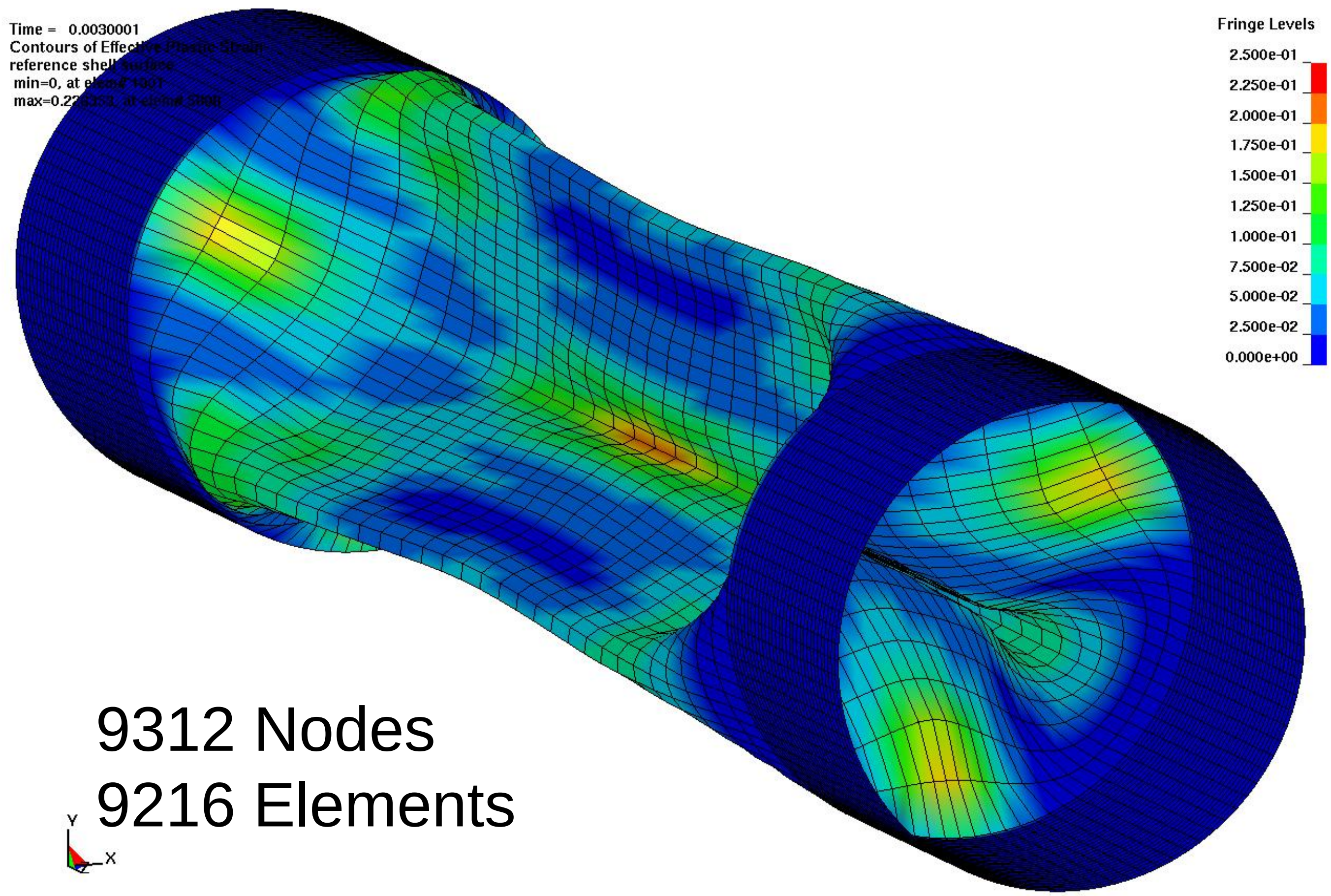
*(**No buckling or plastic deformation** is observed.)*

22800 Elements.
Linear interpolation.

Reference solution courtesy of Ted Belytschko

Short Cylinder Experiment

Effective Plastic Strain at Mid-Surface
Production Belytschko-Tsay Element
Linear C^0 Basis Functions

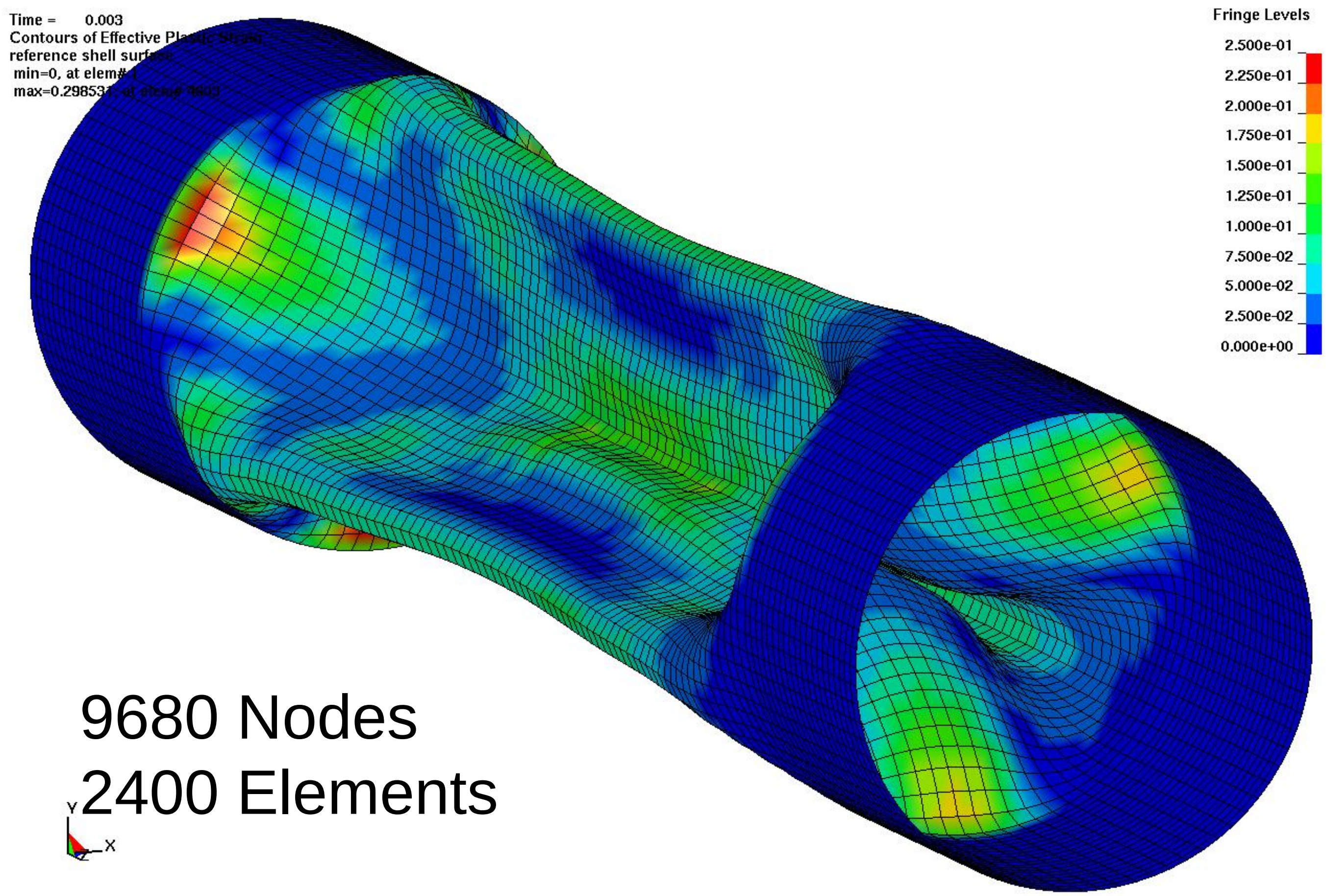


9312 Nodes
9216 Elements

Short Cylinder Experiment

Effective Plastic Strain at Mid-Surface

Quadratic C^0 Basis Functions; IFORM=3



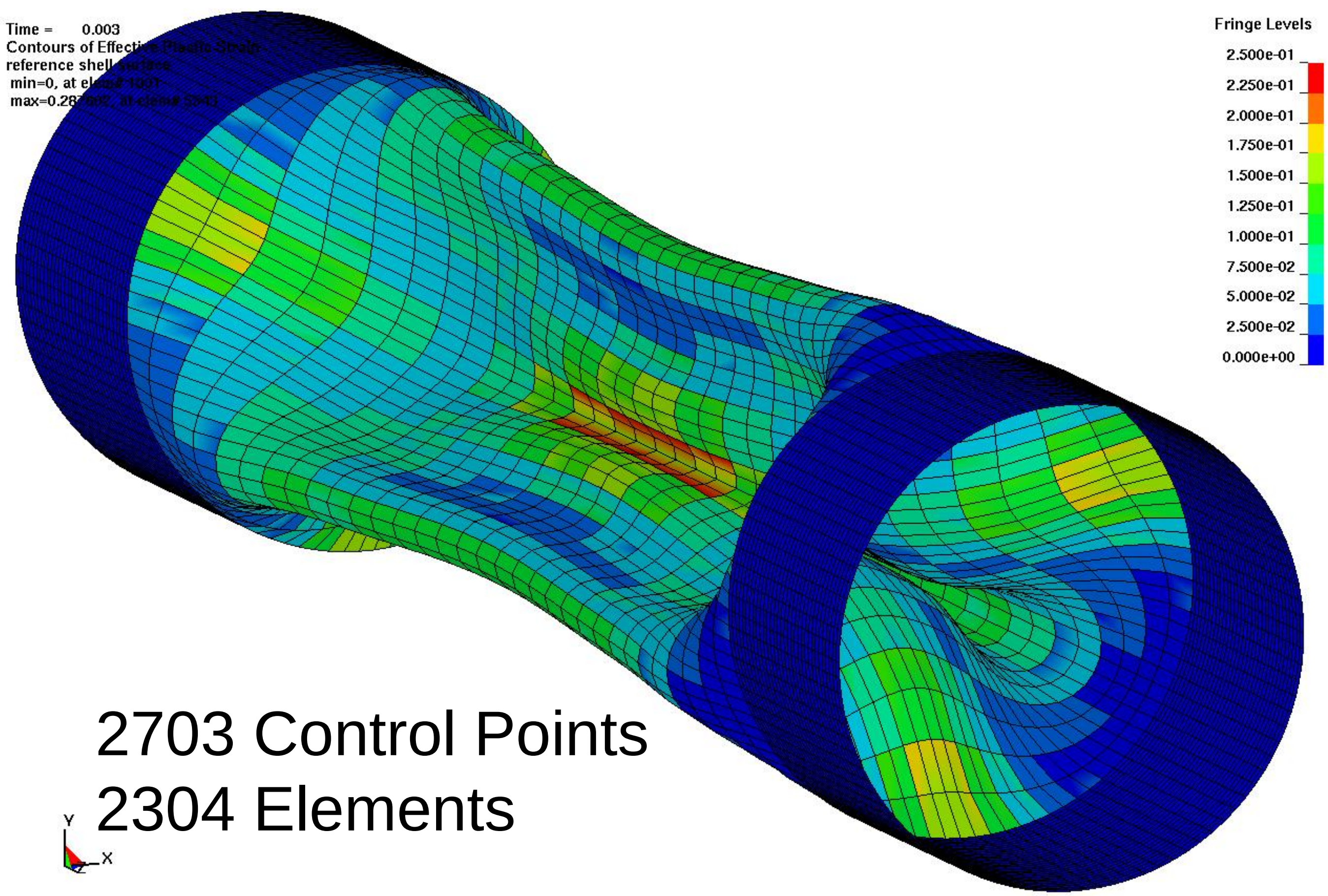
9680 Nodes

2400 Elements

Short Cylinder Experiment

Effective Plastic Strain at Mid-Surface

Quadratic C^1 NURBS; IFORM=0



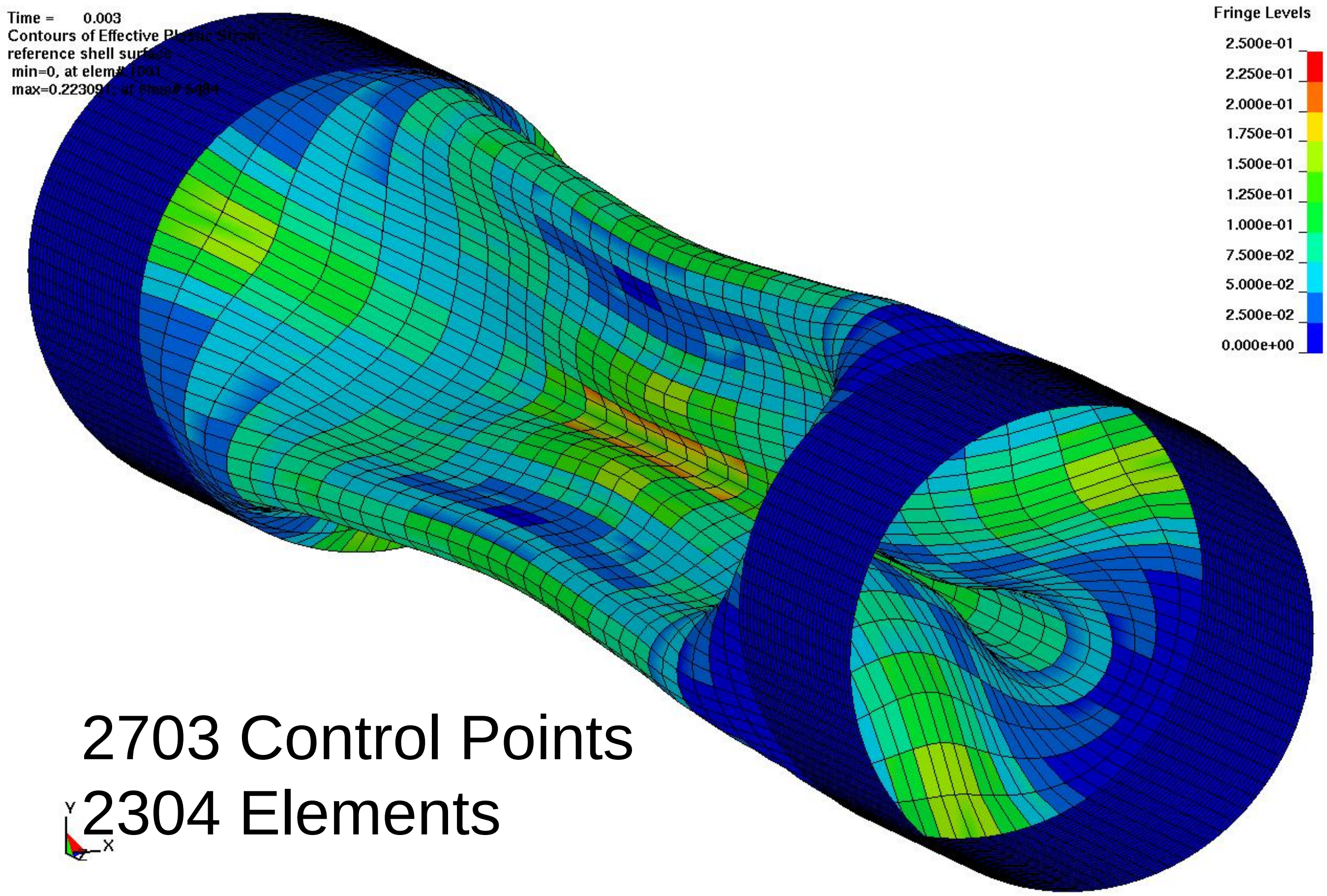
2703 Control Points

2304 Elements

Short Cylinder Experiment

Effective Plastic Strain at Mid-Surface

Quadratic C^1 NURBS; IFORM=3



2703 Control Points

2304 Elements

User-Defined Element Speed

- No performance data presented because:
  - Current implementation is for research, not production.
  - Anticipate future research to improve performance just a 4-node shells improved to current B-T speed.
- User-defined element speed is:
  - Fast enough for implicit.
  - Slow for explicit.

Expected CPU Cost Scaling

- User-defined element overhead:
  - Generality of # nodes and # integ. points.
  - Interpolation elements for contact.
  - Interpolation nodes for force distribution.
- Belytschko-Tsay: 1 point integration & 4 nodes.
- User-defined (U-D) element: G integration points &N nodes.
- U-D/Belytschko-Tsay cost:
  - Cost ratio ~ (G/1)(N/4) ignoring matl. model.
  - Example: 9-node C^0 Lagrangian cost ratio = 81/4 ~ 20
  - Measured: 62/1.5 ~40
- Expect production code to be at least twice as fast as user-defined element without advances in formulation.

Current CPU Costs Due to Shell Formulation

- True curved element formulations.
- Full integration -- loss of simplifications associated with 1-point integration.
  - Integration is area of current research.
  - Early results indicate possible dramatic speed improvements.
- IFORM=2 & 3 have terms involving derivatives of the normal vector (require second derivatives of basis functions).

Summary & Conclusions

- Ultimate goal: direct CAD to analysis.
- Isogeometric elements available in LS-DYNA via user-defined elements.
  - Explicit.
  - Implicit.
  - Eigenvalue analysis.
- Preliminary results suggest that:
  - Higher order accuracy without dt penalty of Lagrangian shape functions.
  - Excellent eigenvalue accuracy.
  - Shell elements without rotational DOF show promise.
  - Robust large deformation performance.