A SIMPLE CORRECTION TO THE FIRST ORDER SHEAR DEFORMATION SHELL FINITE ELEMENT FORMULATIONS

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ABSTRACT

The present work concentrates on the development of correct representation of the transverse shear strains and stresses in Mindlin type displacement based shell finite elements. The formulation utilizes the robust standard first order shear deformation shell finite element for implementation of the proposed representation of the transverse shear stresses and strains. In this manner the need for the shear correction factor is eliminated. In addition, modification to any existing shell finite element for the correct representation of transverse shear quantities is minimal. Some modifications to correct Mindlin type elements are presented in the literature. These modifications correct the distribution of the transverse shear stresses only and use the constant transverse shear strains through the thickness. As compared to the above, the present formulation uses the correct distribution and is consistent for both transverse shear stresses as well as transverse shear strains.

Keyword: shell finite element, transverse shear stresses and strains, higher order shell theory, shear correction factor.

INTRODUCTION

One of the major disadvantages of the first order shear deformation shell theories is that although they account for the transverse shear they cannot correctly represent its through-thickness distribution [1]. Nevertheless, their ability to accurately predict the overall shell behavior and their relative simplicity makes them the basis for most shell elements utilized in the finite element codes nowadays. The first order shell elements are usually capable of also producing good results for the in-plane strain and stress distribution but their formulation results in constant transverse shear strains as opposed to the realistic parabolic distribution. As a result the traction conditions at the shell surfaces are violated. They also require shear correction coefficients to correct the corresponding strain energy terms and these coefficients are problem dependent and are not always easy to determine. Numerous efforts have been made to overcome this disadvantage of the first order formulation most of which result in a higher order shear deformation theory (e.g. see Pandya and Kant [2], Reddy [3], Ha [4], Noor et al. [5]). An efficient remedy for the transverse shear inconsistency is implemented in the finite element codes ABAQUS and MSC/NASTRAN and described in Chapter 3.6.8 of [6] and Chapter 6.5 of [7]. It is based on the stress and moment equilibrium equations and results in a parabolic through thickness distribution for the transverse shear stresses. However, in this formulation the transverse shear strains are still constant through the shell thickness and

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therefore in its implementation the shear correction factors are still required in the strain energy terms.

The herein-presented approach for treating the transverse shear strains and stresses in homogeneous shells results in parabolic distribution for both strains and stresses and, therefore, it eliminates the need of any shear correction factors. It requires only minor changes in the first order shear deformation formulation and totally preserves its efficiency. It is applicable to any displacement-based formulation and extremely easy to implement in any first order shell finite element.

THEORETICAL FORMULATION

The present formulation starts with the third order displacement field

$$\begin{aligned} u &= u_0 + \theta_y z + \varphi_y z^2 + \psi_y z^3 \\ v &= v_0 - \theta_x z - \varphi_x z^2 - \psi_x z^3 \\ w &= w_0 \end{aligned} \quad , \qquad (1)$$

where u and v are the in-plane, and w is the transverse displacement component. Here u_0, v_0, and w_0 are the reference surface linear displacements along the coordinate axes x, y, and z respectively. θ_i, i = 1,2 are the reference surface rotations, and φ_i and ψ_i are the higher order terms in the displacement polynomial expansion. z is the coordinate along the z-axis normal to the shell reference surface.

Using the strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (2)$$

the components of the strain vector corresponding to the displacement field (1) are:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_0}{\partial x} + \frac{\partial \theta_y}{\partial x} z + \frac{\partial \varphi_y}{\partial x} z^2 + \frac{\partial \psi_y}{\partial x} z^3 \\ \varepsilon_y &= \frac{\partial v_0}{\partial y} - \frac{\partial \theta_x}{\partial y} z - \frac{\partial \varphi_x}{\partial y} z^2 - \frac{\partial \psi_x}{\partial y} z^3 \\ 2\varepsilon_{xy} &= \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) z + \left(\frac{\partial \varphi_y}{\partial y} - \frac{\partial \varphi_x}{\partial x}\right) z^2 + \left(\frac{\partial \psi_y}{\partial y} - \frac{\partial \psi_x}{\partial x}\right) z^3 \qquad (3) \\ 2\varepsilon_{yz} &= \left(\frac{\partial w_0}{\partial y} - \theta_x\right) - 2\varphi_x z - 3\psi_x z^2 \\ 2\varepsilon_{xz} &= \left(\frac{\partial w_0}{\partial x} + \theta_y\right) + 2\varphi_y z + 3\psi_y z^2 . \end{aligned}$$

Vanishing of the transverse shear stresses at the top and bottom shell surfaces, $\sigma_{yz}\left(\pm \frac{h}{2}\right) = \sigma_{xz}\left(\pm \frac{h}{2}\right) = 0$, makes the corresponding strains there zero, which yields:

$$\varphi_x \equiv \varphi_y \equiv 0$$

$$3\psi_x = \frac{4}{h^2}\left(\frac{\partial w_0}{\partial y} - \theta_x\right); \quad 3\psi_y = -\frac{4}{h^2}\left(\frac{\partial w_0}{\partial x} + \theta_y\right). \qquad (4)$$

Here h is the shell thickness. Using these relationships the displacement field, Eq. (1), and the strain expressions, Eq. (3), simplify into:

$$\begin{aligned} u &= u_0 + \theta_y z + \psi_y z^3 \\ v &= v_0 - \theta_x z - \psi_x z^3 \\ w &= w_0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_0}{\partial x} + \frac{\partial \theta_y}{\partial x} z + \frac{\partial \psi_y}{\partial x} z^3 \\ \varepsilon_y &= \frac{\partial v_0}{\partial y} - \frac{\partial \theta_x}{\partial y} z - \frac{\partial \psi_x}{\partial y} z^3 \\ 2\varepsilon_{xy} &= \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) z + \left(\frac{\partial \psi_y}{\partial y} - \frac{\partial \psi_x}{\partial x}\right) z^3 \\ 2\varepsilon_{yz} &= \left(\frac{\partial w_0}{\partial y} - \theta_x\right)\left(1 - \frac{4}{h^2} z^2\right) \\ 2\varepsilon_{xz} &= \left(\frac{\partial w_0}{\partial x} + \theta_y\right)\left(1 - \frac{4}{h^2} z^2\right). \end{aligned} \tag{6}$$

Let us now assume that the first two terms in the ε_x, ε_y, and ε_{xy} expressions in Eq. (6) represent the through-thickness distribution of the in-plane strains with enough accuracy. This means that we can neglect the contribution of the derivatives of ψ_x and ψ_x with respect to x and y and simplify the strain relations as follow:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_0}{\partial x} + \frac{\partial \theta_y}{\partial x} z \\ \varepsilon_y &= \frac{\partial v_0}{\partial y} - \frac{\partial \theta_x}{\partial y} z \\ 2\varepsilon_{xy} &= \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) z \\ 2\varepsilon_{yz} &= \left(\frac{\partial w_0}{\partial y} - \theta_x\right)\left(1 - \frac{4}{h^2} z^2\right) \\ 2\varepsilon_{xz} &= \left(\frac{\partial w_0}{\partial x} + \theta_y\right)\left(1 - \frac{4}{h^2} z^2\right). \end{aligned} \tag{7}$$

These expressions are identical to the strain expressions from the first order shear deformation displacement field

$$\begin{aligned} u &= u_0 + \theta_y z \\ v &= v_0 - \theta_x z \ , \\ w &= w_0 \end{aligned} \tag{8}$$

except for the transverse strain expressions. They are different from the usual relations, $2\varepsilon_{yz} = \frac{\partial w_0}{\partial y} - \theta_x$, $2\varepsilon_{xz} = \frac{\partial w_0}{\partial x} + \theta_y$, and result in parabolic through-thickness distribution for the transverse shear strains.

The strain expressions, Eq. (7), and their corresponding displacement field, Eq. (8), define the present formulation, which is actually a first order shear deformation shell formulation with corrected transverse shear. It is evident that it results in a parabolic distribution of the transverse shear strains and satisfies the zero transverse shear stresses requirements at the shell surfaces. As seen, it also requires insignificant modifications to be implemented in existing displacement based first order shell elements.

Now, let us compare the above formulation with the approach in [6] and [7]. The latter approach is based on the equilibrium equations:

$$\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \sigma_x}{\partial x} = 0; \quad V_x + \frac{\partial M_x}{\partial x} = 0. \tag{9}$$

The in-plane normal stress, σ_x, can be expressed through the bending moment M_x

$$\sigma_x = \frac{M_x}{I} z; \quad \frac{\partial \sigma_x}{\partial x} = \frac{12z}{h^3} \cdot \frac{\partial M_x}{\partial x} = -\frac{12V_x z}{h^3}. \tag{10}$$

In the last expression it is assumed that the shell thickness does not vary (or vary slowly) with position along the shell. Note that in the herein-presented formulation this assumption is not required.

Substituting the second equation in (10) into the first relation in (9) and integrating yields

$$\tau_{xz} = C_1 - \frac{6V_x}{h^3} z^2. \tag{11}$$

At $z = \pm \frac{h}{2}$, $\tau_{xz} = 0$ and $C_1 = \frac{3V_x}{2h}$. Then for the transverse shear stress we get

$$\tau_{xz} = \frac{3V_x}{2h}\left(1 - \frac{4}{h^2} z^2\right). \tag{12}$$

Comparing this expression with the last relation in (7) we see that they are very similar. We know that for a homogeneous shell $\frac{3V_x}{2h}$ gives the maximum value of the transverse shear stress τ_{xz}. Obviously $\frac{\partial w_0}{\partial x} + \theta_y$ has similar meaning for the transverse shear strain $2\varepsilon_{xz}$. Therefore, it is expected that both approaches would give the same results for the transverse shear stresses. Note that if the transverse strains in [6] and [7] are calculated from the strain–displacement relations, Eq. (2), there will be an inconsistency between the transverse strains and the transverse stresses, and the approach will still require a shear correction factor in the strain energy expression. This is not the case with the present formulation.

EXAMPLE PROBLEMS

To illustrate the performance of the present formulation and compare it with results from other approaches it is implemented in the explicit finite element code DYNA3D. The formulation of the Belytschko-Lin-Tsay[8] shell element is changed to reflect the present approach. Two models are investigated and the results acquired using the present approach are compared with results from other solution approaches and with previously published results.

First, a simple model is constructed and solved: a strip of length 300 *mm*, width 5 *mm*, and height 2 *mm* is clamped at both ends and subjected to uniform distributed vertical load of magnitude 10 *kPa*. The material is steel with E = 207 *GPa*, $\nu = 0.32$, and density $\rho = 7.83\times10^3$ kg/m^3. Results for transverse shear stress and strain, and normal stress are presented in Figures 1-3 corresponding to two different sections of the strip considered. Stresses and strains are collected at section with coordinate x=2.5 mm and 52.5 mm. Figure 1 presents predictions for the transverse shear stresses of the present approach. The predictions are compared with results obtained from the closed form elasticity solution, results from the finite element code ABAQUS based on the approach described in[6], and results from the original first order shear deformable formulation (FOSDT). As seen the FOSDT results in constant transverse shear stresses through the thickness and the rest of the results coincide very well with each other. Figure 2 presents the results for the transverse shear strains, γ_{xz}. As seen both ABAQUS and the FOSDT result in constant transverse shear strains through the thickness, which are incorrect. However, the transverse shear strains predicted with the present approach agree very well with that obtained from the elasticity solution. The difference between the values from ABAQUS and FOSDT is due to the fact that a shear correction factor of 1 is used in the FOSDT calculations while it is determined automatically in ABAQUS. From Figures 1 and 2 the inconsistency between the distribution of transverse shear stresses and strains can be observed (parabolic for stresses, however, linear for strains). Finally, Figure 3 shows the in-plane normal stresses, σ_x, distribution at 3 different sections along the strip. As seen the corrected transverse shear does not influence the in-plane normal strains, which provide further confidence in the developed corrections.

Second, a model investigating the axial buckling of a cylindrical composite shell is taken from the study of Anastasiadis et al. [9]. Results are acquired using three different approaches: a standard approach for layered shells based on constant transverse shear strains through the shell thickness (Approach 1); an approach for layered shells based on the differential equations of equilibrium (as described in [6] and [7] and denoted with “Approach 2”); and the present approach. The model consists of a composite cylinder fixed at both ends, which has a radius of 0.1905 m and thickness of 12.7 mm. The shell consists of orthotropic boron/epoxy layers with the following material properties: E_{11} = 206.8 GPa, $E_{22} = E_{33}$ = 18.62 GPa, $\nu_{12} = \nu_{13} = 0.21$, $\nu_{23} = 0.45$, $G_{12} = G_{13}$ = 4.482 GPa, G_{23} = 2.551 GPa. It is axially loaded and the buckling value of the load is reported. For L/R = 2 several stacking sequences are investigated and the results are presented in table 1.

Table 1. Critical axial compression in N/m $\times 10^{-6}$

Stacking Sequence	HOSD from [9]	Approach 1	Approach 2	Present
$(0°,90°,0°)_s$	17.82	20.4	18.0	18.2
$(90°,0°,90°)_s$	14.85	16.1	14.5	15.5
$(-45°,45°,-45°)_s$	17.08	20.0	17.3	18.2
$(45°,45°,-45°)_s$	13.35	14.2	13.0	13.2

As seen the present results are very good compared to the values in [9] calculated using a higher order shear deformable shell formulation. Fig. 4 shows the force vs. displacement relation acquired in the analysis for a standard FOSDT shell and for the same shell with the herein-proposed modification.

CONCLUSIONS

A correction to the first order shear deformable shell theory is proposed, which results in parabolic through-thickness distribution of the transverse shear strains and stresses. It eliminates the need for shear correction factors in the first order theory. The approach is applicable to all displacement based first order formulations and is simple and extremely easy to implement in any standard shell finite element. As compared to the other modifications presented in the literature, in which correction is made in the distribution through the thickness of the transverse shear stresses only, the proposed modification is more consistent. The present formulation uses the correct distribution and is consistent for both transverse shear stresses as well as transverse shear strains.

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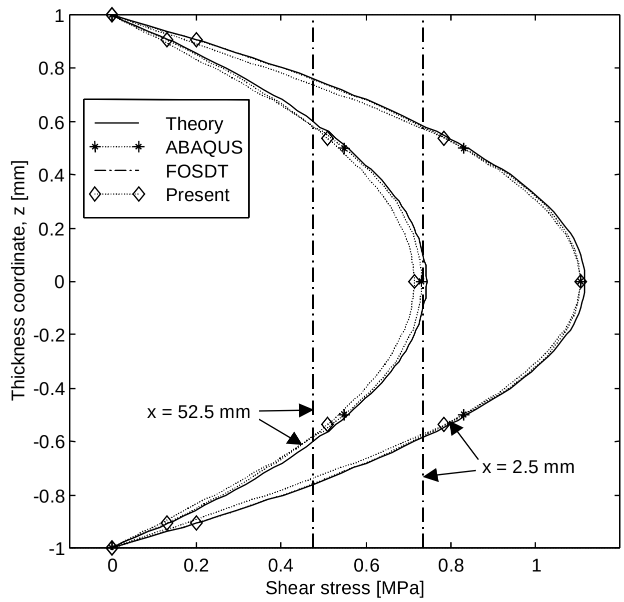

Figure 1 Transverse Shear Stress, σ_{xz}, Distribution through the Shell Thickness in Example Problem 1

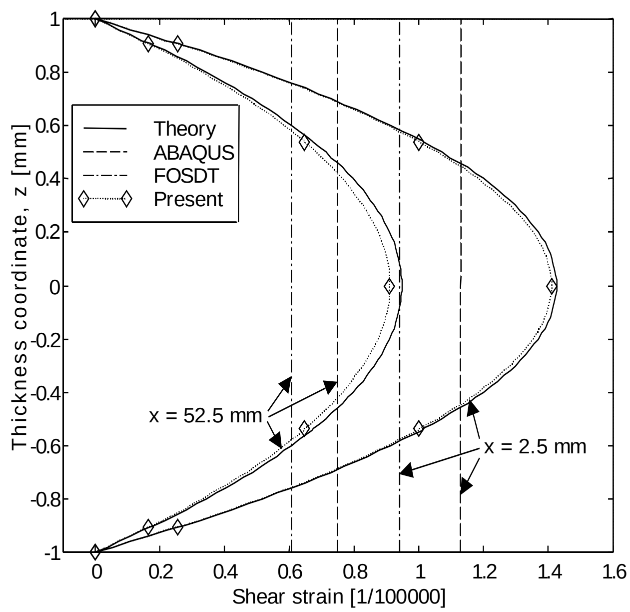

Figure 2 Transverse Shear Strain, γ_{xz}, Distribution through the Shell Thickness in Example Problem 1

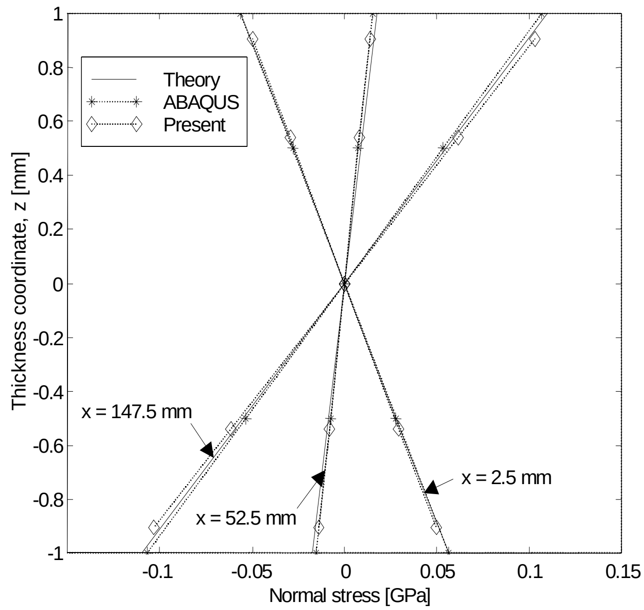

Figure 3 In-plane Normal Stress, σ_x, Distribution through the Shell Thickness in Example Problem 1

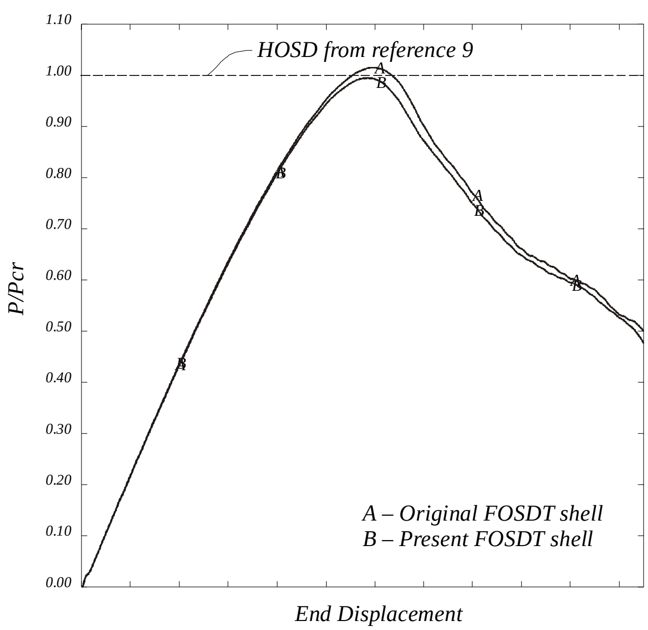

Figure 4 Force vs. End Displacement Curve Normalized with Respect to the Critical Force Value in Example Problem 2