# Strain-softening in continuum damage models: Investigation of MAT_058

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## Abstract

Composite materials are of increasing interest to automotive and aviation industry due to their high strength and stiffness. Therefore they are commonly used to replace metallic materials. However their mechanical behaviour is complex, especially when damage is considered. Composite damage leads to degradation of material properties which results in behaviour known as strain-softening.
An implementation of strain-softening in numerical codes, such as LS-DYNA®, leads to mesh sensitivity of results and therefore those models are not reliable. The user of damage models with strain-softening needs a good understanding of those material models to evaluate results critically.
This work aims to provide an insight on strain-softening behaviour in a mathematical sense and its consequences on numerical codes. An analytical solution is derived for a one-dimensional dynamic bar problem which allows a direct comparison with numerical results. It was found that deformation localises in an area which is governed by the chosen element size and therefore causes mesh sensitivity. Strain grows infinitely in the strain-softening area with a simultaneous drop of stress. Outside the strain-softening area the problem unloads elastically. The dissipated energy tends to vanish.

**Keywords:** material stability, wave equation, deformation localisation, imaginary wave speed.

## 1 Introduction

Composite materials are important materials in automotive and aerospace industry; however, their design is complicated. Finite element (FE) codes are used to support the design processes of structures.
A realistic representation of composite material behaviour requires the description of stiffness degeneration due to damage. The continuum damage mechanics (CDM) approach enables the inclusion of damage description into well-known constitutive equations. Nonetheless, the CDM approach is limited in its applicability for strain-softening materials. Material strain-softening occurs when the stress-strain relationship has a negative slope outside the linear elastic domain. In this case the CDM approach ceases to be mathematically meaningful.
The CDM approach is mathematically unstable when strain-softening behaviour is included. This is the case for composites. In case of strain-softening the initial value problem is ill-posed and leads to severe numerical problems in FE codes. Numerical results in the strain-softening zone are lacking physical meaning and depend highly on the mesh discretisation of the problem.
The dynamic FE code LS-DYNA® is considered as "state-of-the-art" FE software and is widely used in automotive and aerospace technology. It provides a CDM model for composites (MAT_058) which follows the suggestions of Matzenmiller et al. [1]. This model is the only composite CDM model which is available without acquiring an additional licence.
This work aims to support the understanding of strain-softening behaviour and consequences of strain-softening when implemented in FE codes. Of special interest will be the case of dynamic strain-softening. An analytical solution of the dynamic strain-softening problem derived by Bažant and Belytschko [2] will be discussed. These results will be compared with solutions by LS-DYNA.
Section 2 will focus on the background theory. The origin of the strain-softening problem will be stated and an analytical solution for a dynamic strain-softening problem will be discussed. The operating mode of the composite CDM model in LS-DYNA will be stated. Section 3 will show the methodology used to obtain numerical results. Section 4 will present the results of this study. Analytical and

numerical results will be compared. Especially, the outcomes of strain-softening in FE codes, such as mesh sensitivity of results, will be addressed.

## 2 Conceptual Framework

Realistic modelling of composite material behaviour includes degeneration of material properties due to damage. Damage material models are generally based on the CDM approach. CDM represents microscopic heterogeneous damage on the macroscale. Degeneration of material properties is described by a loss of effective load carrying area designated with a damage variable. This variable can be scalar or tensorial. The expressions for constitutive equations are derived from well-known principals, such as effective stress in conjunction with equivalence hypethesises, such as strain equivalence or energy equivalence.
Despite the need of the CDM approach to describe damage, it can lead to an ill-posed boundary value problem when the tangent stiffness tensor losses its positive-definiteness. This causes partial differential equations (PDEs) in dynamic problems to change their type from hyperbolic to elliptic, or from elliptic to hyperbolic in static problems.

### 2.1 Strain-softening

The development of localised deformation is caused by a process on the material's microscale. The microscopic behaviour is governed by the occurrence, growth and interaction of cracks and voids which finally lead to complete fracture on the macroscale. In the following another definition of localisation will be used which enables the use of constitutive equations. This one was made by Rudnicki and Rice [3]: "localization can be understood as instability in the macroscopic constitutive description of inelastic deformation of the material". The instability allows the constitutive equations of an originally homogeneous material to reach a bifurcation point where the non-uniform deformation localises. Outside this localisation zone the material continues to be in equilibrium. [3]
If a material with a "conventional, rate-independent simple constitutive equation" is implemented in finite element codes, obtained results will be dependent on the mesh discretisation in the inelastic area. This dependence is a result of the governing differential equation being ill-posed.
The loss of well-posedness is caused by a type change of partial differential equations. In static problems the PDEs change from an elliptic to a hyperbolic type. This is called loss of ellipticity. In dynamic problems the PDEs change from a hyperbolic to an elliptic type.
A material is considered to be stable and stay in equilibrium when the inner product of stress rate $\dot{\sigma}_{ij}$ and strain rate $\dot{\varepsilon}_{ij}$ is positive. The criterion is also called general bifurcation criterion [4]. This is true as long as the stress-strain relationship of the material has a positive slope.

$$\dot{\varepsilon}_{ij}\dot{\sigma}_{ij} > 0 \tag{1}$$

The rate format of constitutive equations will be used furthermore to contain a piecewise linear relationship between stress and strain. The stress rate can be expressed in dependence on the strain rate. The connection between stress rate and strain rate is made by the material tangent stiffness tensor $D_{ijkl}$. $D_{ijkl}$ is assumed to describe orthotropy as most complex material behaviour. Therefore, the tangent stiffness tensor is symmetric: $D_{ijkl} = D_{klij}$.

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} \tag{2}$$

Therefore, the inequality in (1) reads:

$$\dot{\varepsilon}_{ij}D_{ijkl}\dot{\varepsilon}_{kl} > 0 \tag{3}$$

The material becomes unstable when the material reaches its limiting point. This point occurs when the condition in (3) is violated. The condition in (4) is called bifurcation criterion.

$$\dot{\varepsilon}_{ij}D_{ijkl}\dot{\varepsilon}_{kl} = 0 \tag{4}$$

This condition appears when the tangent stiffness tensor has a singularity. The tangent stiffness tensor loses its positive-definiteness when:

$$\det\left(D_{ijkl}\right) = 0 \tag{5}$$

### 2.2 Analytical Solution of Dynamic Strain-softening

The dynamic strain-softening problem is highly nonlinear. However, Bažant and Belytschko [2] were able to derive an analytical solution for a one-dimensional wave propagation problem.
Fig. 1 shows the stress-strain curve of a general strain-softening material considered by Bažant and Belytschko [2]. The linear elastic area is shown between the points O and P. The stiffness is given by the Young's modulus $E$. The maximum strength $f_t^{'}$ is reached for the plastic strain $\varepsilon_p$. The curve in the strain-softening area (area between points P and F) is given by the function $F(\varepsilon)$. The slope of this curve, $F'(\varepsilon)$, is negative. $F(\varepsilon)$ reaches a zero stress value for a finite strain $\varepsilon$ or an asymptotic strain $\varepsilon \to \infty$. Unloading $(\dot{\varepsilon} < 0)$ and reloading $(\dot{\varepsilon} \geq 0)$ is considered to be elastic and happens with the Young's modulus $E$ of the linear elastic area.

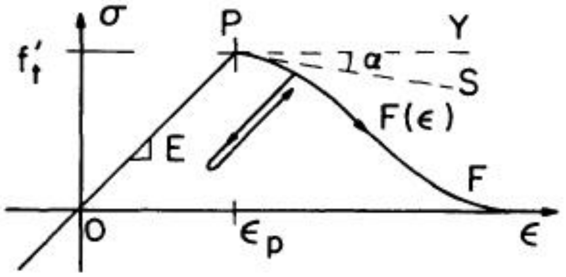


*Fig. 1: Stress-strain diagram of strain-softening material* [2]

The geometry and loading conditions are shown in Fig. 2. The bar has a length of $2L$ and a density $\rho$ per unit length. The bar is loaded at both ends with a constant velocity $v$ which acts in opposite directions. Therefore, tensile waves are generated at the two ends which travel towards the centre of the bar. The longitudinal coordinate $x$ is measured from the bar's centre.

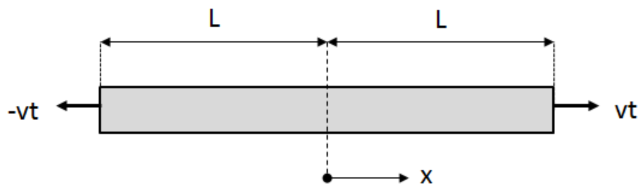


*Fig. 2: Geometry and loading of strain-softening bar* [2]

Two step waves are generated in the bar. One wave travels from the right boundary in the negative $x$-direction. The other wave travels from the left boundary in the positive $x$-direction. The two step waves of constant strain travel to the centre of the bar and meet at $x = 0$ for the time $t = L / c_e$. When the two waves meet strain doubles instantaneously at the centre of the bar and the midsection enters immediately the strain-softening regime.
The equation of motion is the wave equation. The equation is hyperbolic for real wave speeds. Before the onset of strain-softening the problem is governed by this standard equation with the elastic wave speed $c_e = \sqrt{E / \rho}$.

$$c_e^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{7}$$

The longitudinal displacement function in the linear elastic domain is derived from appropriate initial and boundary conditions.

$$u(x,t) = -v\left\langle t - \frac{x+L}{c_e} \right\rangle + v\left\langle t + \frac{x-L}{c_e} \right\rangle \quad (8)$$

The expressions in the brackets $\langle \bullet \rangle$ need to be positive. Therefore, the following definitions are used: $\langle A \rangle = A$ if $A > 0$ and $\langle A \rangle = 0$ if $A \leq 0$.

The according strain function needs to be positive. Therefore, the Heaviside step function $H(\bullet)$ is used.

$$\varepsilon = \frac{\partial u}{\partial x} = \frac{v}{c}\left[ H\left( t - \frac{x+L}{c_e} \right) + H\left( t + \frac{x-L}{c_e} \right) \right] \quad (9)$$

The stress caused by the deformation is described with Hooke's law for linear elasticity.

$$\sigma = E\varepsilon \quad (10)$$

The strain in the bar is assumed to fulfil the condition: $\varepsilon_p / 2 < \varepsilon \leq \varepsilon_p$. The assumption of linear elasticity only holds for $t < L / c_e$. The midsection enters immediately the strain-softening regime for $t = L / c_e$. Therefore, the solution for the displacement $u(x,t)$ in Equation (8) holds only for $t < L / c_e$.

The slope of the stress-strain curve in the strain-softening domain is $F'(\varepsilon)$. For this domain the slope is negative: $F'(\varepsilon) < 0$. The wave speed becomes imaginary. Therefore, the equation of motion in the strain-softening domain is an elliptic differential equation.

$$c_e^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0 \text{ with } c_e^2 = \frac{F'(\varepsilon)}{\rho} \quad (11)$$

Strain-softening is limited to an area of measure zero around $x = 0$. A discontinuity with a displacement jump develops at $x = 0$. The difference in magnitude is $4v\langle t - L / c_e \rangle$. Strain starts to increase infinitely and stress drops to zero in the strain-softening zone. The rest of the bar starts to unload in an elastic manor.

Strain in the strain-softening domain can be expressed by the Dirac Delta function:

$$\varepsilon = 4v\langle t - L / c_e \rangle \delta(x) \quad (12)$$

The solutions for the strain field outside the strain-softening zone, $t > L / c_e$ and $x < 0$, is:

$$\varepsilon = \frac{v}{c_e}\left[ H\left( t - \frac{x+L}{c_e} \right) + H\left( t + \frac{x-L}{c_e} \right) + 4v\langle t - L / c_e \rangle \delta(x) \right] \quad (13)$$

The solution in 4-7 is symmetric for $x > 0$.

The analytical solution by Bažant and Belytschko [2] was used to derive a comparison between an elastic ($\varepsilon < \varepsilon_p / 2$) and a strain-softening ($\varepsilon_p / 2 < \varepsilon \leq \varepsilon_p$) wave propagation problem. The solutions (see Fig. 3) are compared for the time $t = 3/2 \cdot L / c_e$. Displacement, strain and stress profiles are plotted along the length of the bar. The internal energy is plotted for the whole bar in the time interval $0 \leq t \leq 2 \cdot L / c_e$. The character of the strain-softening problem becomes obvious in Fig. 3. Strain-softening occurs in the midsection of the bar ($x = 0$) with a displacement jump. The according strain reaches infinity at $x = 0$ in an area with zero length. The strain in the remaining bar does not double after reflection at $x = 0$ as in the elastic solution. Instead a release wave occurs after reflection. The stress drops to zero at $x = 0$ and unloading occurs after reflection. A further consumption of energy is not possible after $t = L/c_e$ for the strain-softening problem.

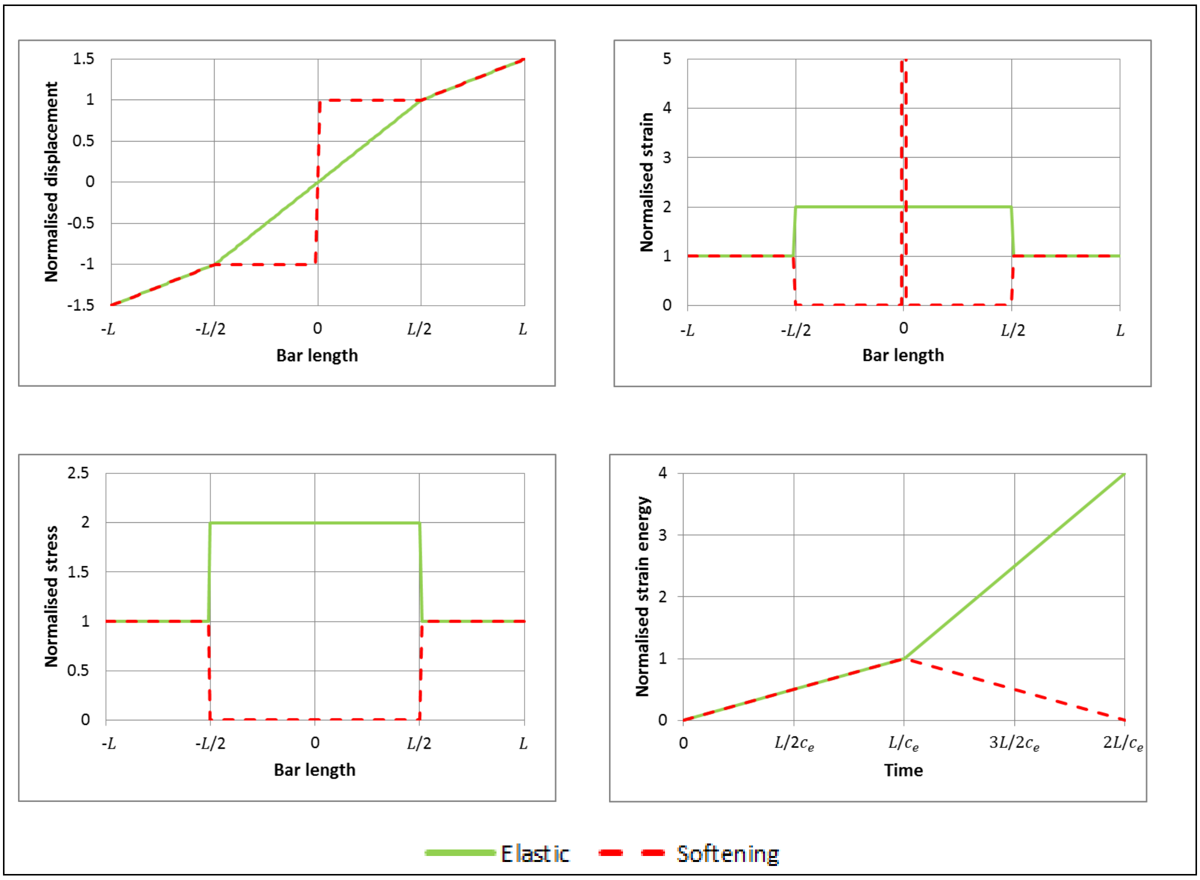


Fig. 3: Analytical solution for displacement, strain and stress for $t = 3/2 \cdot L/c_e$ and internal energy over time

### 2.3 LS-DYNA Continuum Damage Mechanics Model for Composites (MAT_058)

MAT_058 is based on the two-dimensional anisotropic damage model for composites by Matzenmiller et al. [1]. Damage $d$ initiates after the material exceeds the initial elastic area. The expression of damage is exponential:

$$d_i = 1 - \exp\left[ -\frac{1}{m_i e} \left( \frac{\varepsilon_i}{\varepsilon_{fi}} \right)^{m_i} \right] \tag{14}$$

The index $i$ identifies the considered principal direction. The constant $e$ is the Euler's Number. $m$ is the strain-softening parameter which can be assumed to be hardwired in MAT_058 as equal to $10$ for composites. A parameter $m = 10$ indicates brittle behaviour. $\varepsilon_{fi}$ is the failure strain in the according principal axis at which damage first initiates:

$$\varepsilon_{f1} = \frac{X_{t,c}}{E_1} \tag{15}$$

$$\varepsilon_{f2} = \frac{Y_{t,c}}{E_2} \tag{16}$$

$$\varepsilon_{f12} = \frac{S}{G} \tag{17}$$

$X$, $Y$ and $S$ are the materials strengths in longitudinal, transverse and shear direction for tension and compression. $E_1$, $E_2$ and $G$ are the Young's moduli in longitudinal or transverse direction and the shear modulus.

## 3 Method

The strain-softening bar has an overall length of $2L$ and a squared unit cross-section. The origin of the coordinate system lies in the centre of the bar. The longitudinal direction is denoted with $x$.
MAT_058 demands as input the material properties in longitudinal and transverse direction. The conducted test is a uniaxial tensile test. The longitudinal principal direction coincides with the longitudinal coordinate $x$. Tensile loads will be applied in longitudinal direction $x$.
Bažant and Belytschko [2] describe a uniaxial bar in their analytical approach to the strain-softening problem. However, MAT_058 is implemented with components comprised of shell elements. Therefore, all degrees of freedom except for the longitudinal direction are restricted to comply with the uniaxial theory.
In the previously published literature it is stated that strain-softening behaviour causes a sensitivity of results on the chosen mesh discretisation. Therefore, the same experiment will be conducted with different element numbers along the bar. An odd number of elements will be used to ensure that strain-softening only occurs in a single element which is positioned in the centre of the bar. The following numbers of elements were used: 5, 11, 21, 31 and 101.
The numerical experiments are meant to show strain-softening when the loading waves meet in the centre of the bar. Hence, the bar will obey the elastic solution for $t = L/c_e$. The strain-softening solution will be valid for $t > L/c_e$. Strain-softening has to occur instantaneously at $x = 0$ when both loading waves meet. Therefore, the tensile loading $v$ in the elastic area has to follow Bažant's and Belytschko's [2] suggestion:

$$\frac{\varepsilon_f}{2} \le \frac{v}{c_e} < \varepsilon_f \tag{18}$$

## 4 Results

Experiments were conducted with the CDM model for composites in LS-DYNA (MAT_058). The uniaxial loading tests show a strong sensitivity of results on the mesh discretisation in the strain-softening domain.
The results of different mesh discretisations are compared for $t = 3/2 \cdot L/c_e$ with analytical solutions. At this time the waves from the left and right have both travelled $3/2$ of the bar. Therefore the area around the bar's centre is strain-softening and the bar's edges are still governed by the elastic solution.
All numerical results will be compared with the analytical results of the strain-softening solution for $t = 3/2 \cdot L/c_e$. At this time the bar area of $-L/2 \le x \le L/2$ obeys the strain-softening solution. The bar's edge areas, $-L \le x < -L/2$ and $L/2 < x < L$ are still governed by the elastic solution.
Fig. 4 shows the longitudinal displacement along the bar for different element discretisations at $t = 3/2 \cdot L/c_e$. A strong sensitivity of results is visible in the strain-softening area. The results approach the analytical solution with increasing number of elements. A sensitivity of results is not visible in the remaining elastic area. However, the accuracy of results rises slightly with finer mesh spacing.
The results in Fig. 4 fit the predicted strain-softening behaviour for longitudinal displacement well. Analytically a displacement jump is predicted for the strain-softening discontinuity with infinitesimal small area at $x = 0$. The bar unloads outside of this area and all displacement accumulates at $x = 0$. This strain-softening behaviour is represented well in the numerical results. The strain-softening area narrows with increasing number of elements as all displacement accumulates in one element. All elements except for the strain-softening element do not show any displacement.

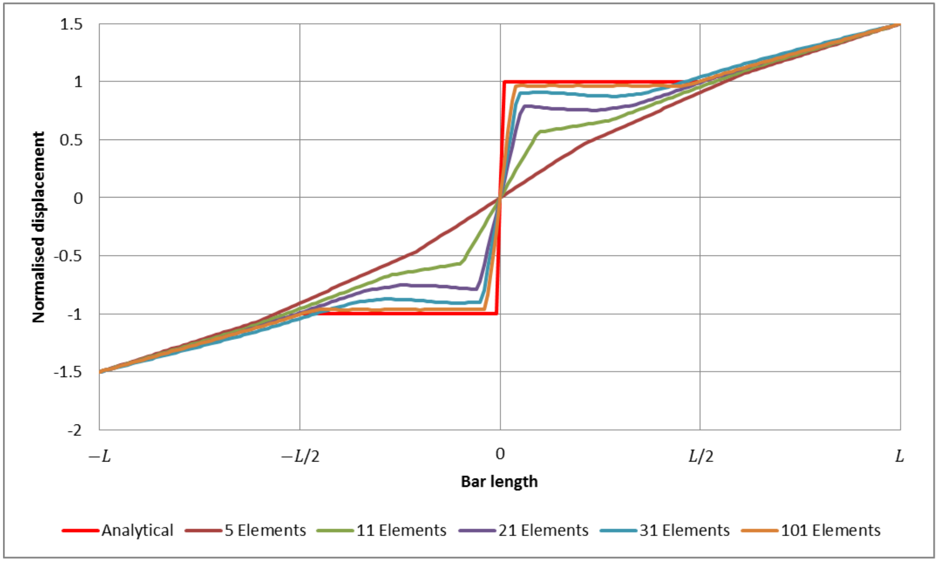


*Fig. 4: Normalised longitudinal displacement of strain-softening bar at* $t = 3/2 \cdot L/c_e$

Results for longitudinal strain along the bar are presented in Fig. 5 for $t = 3/2 \cdot L/c_e$. A strong sensitivity of results on the mesh discretisation is visible. The strain measured in the strain-softening elements increases with increasing mesh refinement. A sensitivity due to mesh discretisation is not visible for the remaining elastic area although accuracy improves with finer mesh spacing.
The numerical results presented in Fig. 5 agree well with the predicted strain-softening behaviour. Strain is supposed to be infinite in the strain-softening discontinuity at $x = 0$. The bar unloads outside the discontinuity and strain gradually drops to zero. The numerical results reflect this strain-softening behaviour well. The strain-softening element in the middle of the bar undergoes intense deformation. Strain increases with increasing element number. Strain in elements outside the strain-softening element gradually drops to zero.

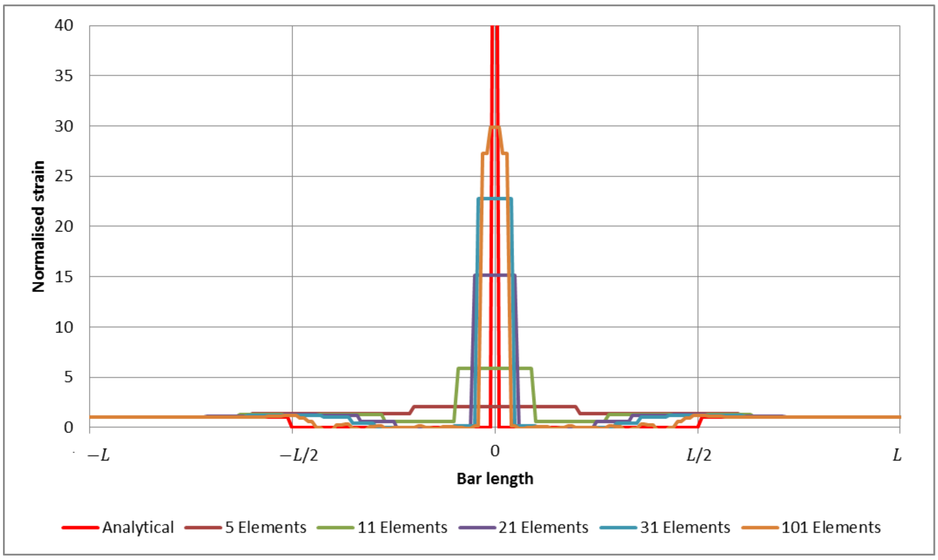


*Fig. 5: Normalised longitudinal strain of strain-softening bar at* $t = 3/2 \cdot L/c_e$

Fig. 6 shows the longitudinal stress development for $t = 3/2 \cdot L/c_e$. A strong sensitivity of results depending on the mesh discretisation is visible for the bar's strain-softening area as well. The results approach the predicted analytical results with increasing number of bar elements. Mesh sensitivity cannot be observed in the remaining elastic area although the accuracy of the results improves with finer mesh spacing.

The numerical results in Fig. 6 are in good agreement with the predicted strain-softening behaviour for longitudinal stress. Due to strain-softening an immediate drop of stress is predicted in the strain-softening area at $x = 0$ and a gradually unloading outside of it between $t = 2 \cdot L / c_e$ and $t = 3/2 \cdot L / c_e$. Numerical results follow these predictions well. The drop of stress to zero becomes more pronounced with decreasing element size.

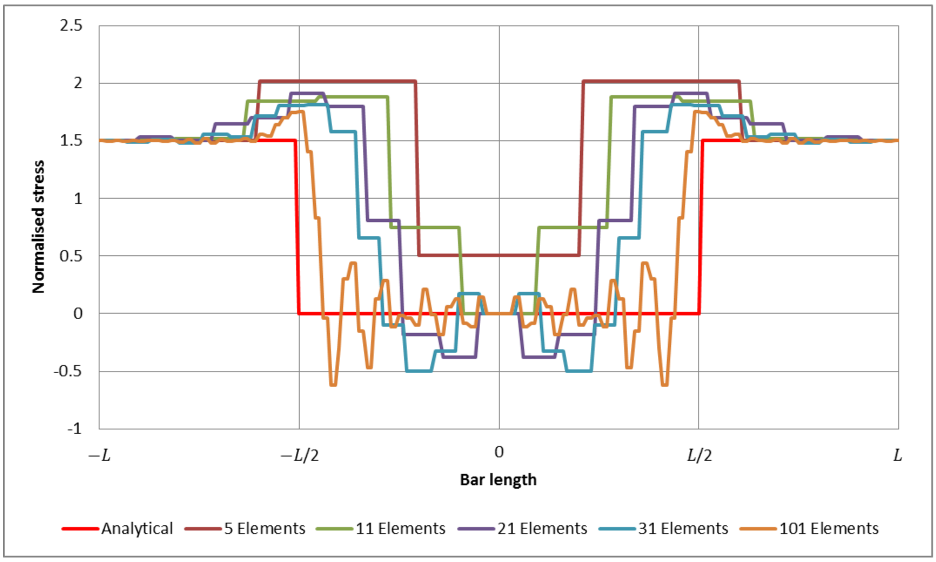


*Fig. 6: Normalised longitudinal stress of strain-softening bar at* $t = 3/2 \cdot L / c_e$

Fig. 7 shows the internal energy consumed by the strain-softening bar in the time interval $0 \le t \le 2 \cdot L / c_e$. The bar is in the elastic domain for $0 \le t \le L / c_e$. A sensitivity of results is not present for this time interval. An improvement in result accuracy can be observed with increasing discretisation fineness only. The bar is in the strain-softening domain for $L / c_e \le t \le 2 \cdot L / c_e$. The results depend strongly on the chosen mesh discretisation in this time interval. The numerical results approach the analytical strain-softening solution with increasing number of elements.
Due to strain-softening behaviour no further energy can be consumed after $t = L / c_e$. The internal energy vanishes gradually due to unloading of the bar after the start of strain-softening.

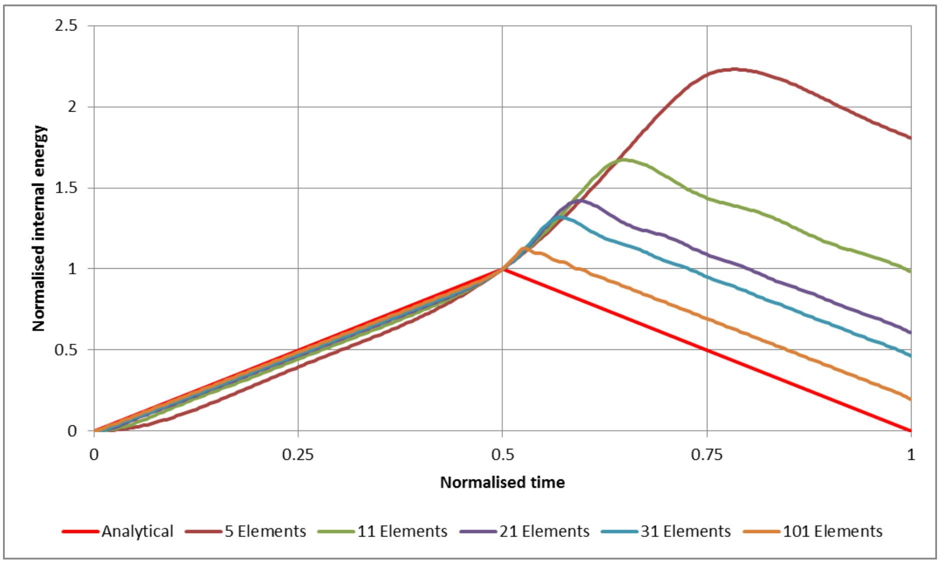


*Fig. 7: Normalised internal energy of strain-softening bar for* $0 \le t \le 2 \cdot L / c_e$

The CDM approach implemented in LS-DYNA (MAT_058) clearly leads to strain-softening and deformation localisation. Displacement, strain, stress and internal energy results show a strong dependence on the chosen mesh discretisation. Strain localises in a single element which is the smallest possible zone in the finite element simulations. Therefore, the size of the localisation zone decreases with increasing number of elements. The CDM approach leads to numerical and physical incorrect results when strain-softening behaviour is present. The numerical results depend strongly on the chosen mesh discretisation.

## 5 Summary

The aim of this work was to gain a better understanding of strain-softening behaviour in CDM models. Of interest were the mathematics of strain-softening and the numerical outcomes. A one-dimensional dynamic strain-softening problem by Bažant and Belytschko [2] was used for the mathematical and numerical investigations. The implementation of the dynamic strain-softening problem was done with a composite CDM model in LS-DYNA (MAT_058).

Strain-softening behaviour in CDM models is supposed to cause mathematical and numerical instabilities and cause physically irrelevant results. It was shown that strain-softening leads to an ill-posed boundary value problem because of ceasing positive-definiteness of the tangent stiffness in the strain-softening area. As a consequence the PDEs change their type from hyperbolic to elliptic which does not fit the initial dynamic equation of motion.

Due to the consideration of Bažant's and Belytschko's [2] one-dimensional dynamic strain-softening bar it was possible to have an insight into the mathematics of strain-softening and the consequences on results, such as displacement, strain, stress and internal energy. The analytical strain-softening problem is limited to an area with zero width. The strain is infinite in the strain-softening domain and stress vanishes. Outside the strain-softening domain strain and stress vanish. Those problems are caused by an imaginary wave speed in the strain-softening domain which forces the PDEs to change from an hyperbolic to an elliptic type. Therefore, the boundary value problem becomes ill-posed.

The analytical problem by Bažant's and Belytschko's [2] was implemented in LS-DYNA using a composite CDM model. It was shown that strain-softening leads to a sensitivity of results to the mesh discretisation. This dependence on the mesh discretisation was observed for displacement, strain, stress and internal energy. Strain localises in the smallest area possible, this is in numerical tests the width of a single element. Results approach the analytical strain-softening solution with increasing fineness of the mesh size.

Users of CDM damage models in FE codes need to be aware of strain-softening behaviour because strain-softening leads to numerically and physically incorrect results. Therefore, the prediction capability of CDM models is limited when applied to structural designs.

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