On Accuracy and Stability of Implicit Time Integration Schemes for Rotating Structures

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Abstract

In the context of finite element analysis, the Newmark time integration scheme is the most commonly used for nonlinear implicit dynamic applications. While it is characterized by unconditional stability and energy conservation, it is also prone to numerical instability when the models are subjected to rotational motion. To this end, LS-DYNA® offers a selection of alternate integration schemes to remedy this deficiency; Bathe, HHT (Hughes-Hilber-Taylor) and FRD (Finite Rotational Dynamics). The intention with this paper is to tentatively discuss these instabilities and investigate to what extent they can be resolved by incorporating more sophisticated schemes.

Introduction

The primary example we have in mind is the design of a turbo fan aero engine. Most analyses relating to the rotors, vibration, stability and interaction with the static structure are analyzed using rotor dynamics models where the details of the rotors are reduced to center line models subject to gyroscopic and rubbing forces. The rotors are connected to models of the static structure of the engine and aeroplane via nonlinear springs. With the advent of supercomputers, detailed finite element models have been constructed to model issues associated with certification. An example is the detailed calculation of a blade off test of an engine. In this context, aircraft engines must demonstrate that they are capable of losing a fan blade at the highest energy and then running down safely without hazarding the aircraft. The engine must subsequently be capable of running freely for up to three hours. The test is both expensive and critical so usually before the test a detailed finite element model is produced and analyzed to give confidence that the test will be successful. The detailed model is run as an explicit analysis and should meet containment criteria and give confidence in the loads for the first few revolutions. Then, rotor dynamics models are constructed and correlated to the results of the test to calculate the rundown and windmilling loads to demonstrate that any loads produced on the aircraft or engine structure are within safe bounds.

Over recent years the finite element matrix equation solution capability has grown with the ability to efficiently use thousands of processors, see for instance Ashcraft et.al. [1]. The question arises as to whether it is feasible to use the full detailed model with implicit time steps to calculate the rundown. Further over the horizon is the prospect of a virtual engine where more physics is added to the model. Crucial to any such analysis is the ability to integrate the equations of motion of a rotor over hundreds of revolutions both accurately and efficiently with ideally large time steps. In a recent study, Kober et.al. [2,3], demonstrate how instabilities arise in a rotating structure using the standard Newmark algorithm and propose ways to deal with it. This paper continues the discussion, and in particular addresses the careful assessment of integration schemes against the required analysis. The paper shows for certain models there are advantages in the FRD scheme which should be explored further.

Implicit Time Integration Schemes

Implicit time integration schemes assume a set of velocities $\boldsymbol{v}_n$ and accelerations $\boldsymbol{a}_n$ at time step n, and given a geometry increment $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_n$ and time step Δt gives expressions for how these are updated to corresponding quantities $\boldsymbol{v}$ and $\boldsymbol{a}$ at time step $n + 1$.

In general, each component of the acceleration, velocity and geometry vectors can be written on the form

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X}_n, \qquad \boldsymbol{X} = \begin{pmatrix} a \\ v \\ x \end{pmatrix}, \qquad \boldsymbol{X}_n = \begin{pmatrix} a_n \\ v_n \\ x_n \end{pmatrix}$$

where $\boldsymbol{A}$ is an integration matrix that depends on the time step and parameters related to the choice of method. The method is *stable* if the spectral radius $\rho(\boldsymbol{A})$, i.e., the maximum eigenvalue of $\boldsymbol{A}$, is not greater than 1. This is a mathematical way to say that the solution of a closed system, with respect to a norm of sufficient strength, stays bounded. The scheme is said to be *unconditionally* stable if the criterion only depends on the numerical parameters and not the time step, a typical characteristic of implicit time integration schemes.

A complementary criterion is that the scheme *converges*, meaning that the numerical solution can be put arbitrarily close to the analytical solution by selecting a sufficiently small time step. With this being said, the main interest is not *if* the scheme converges but rather at which *rate* it does so. Numerical dissipation and dispersion affecting the main frequency content of the analytical solution tend to slow down convergence and should thus be avoided. A good scheme typically shifts these sources of errors towards higher frequencies, something that may even have a stabilizing effect for nonlinear problems.

This section presents the schemes available in LS-DYNA and some of their characteristics, without elaborating on details. For readers inclined to mathematical analyses of the methods we refer to references mentioned below. The integration scheme parameters are collected in the keyword

***CONTROL_IMPLICIT_DYNAMICS**

IMASS	**GAMMA**	**BETA**					**ALPHA**

where only the parameters of interest are included here. For a detailed description we refer to the LS-DYNA Keyword User's Manual [4].

Newmark

The family of Newmark time integration schemes, sometimes referred to as the β-method of Newmark, is by far the most commonly used in structural mechanics and dates back to the works of Nathan Newmark [5]. Mathematically the scheme is given as

$$\boldsymbol{a} = \boldsymbol{a}_n + \frac{\Delta\boldsymbol{x}}{\beta\Delta t^2} - \frac{\boldsymbol{v}_n}{\beta\Delta t} - \frac{1}{2\beta}\boldsymbol{a}_n$$
$$\boldsymbol{v} = \boldsymbol{v}_n + \Delta t(1-\gamma)\boldsymbol{a}_n + \gamma\Delta t\boldsymbol{a}$$

where γ and β are independent parameters and $\Delta\boldsymbol{x}$ is the geometry change between the two time steps. This method is activated in LS-DYNA by **IMASS=1**, **ALPHA=0**, **GAMMA=**γ and **BETA=**β , i.e., the latter two selected as desired. For linear analysis the method is unconditionally stable when

$$\gamma \geq \frac{1}{2}$$
$$\beta \geq \frac{1}{4}\left(\frac{1}{2}+\gamma\right)^2,$$

which in practice can be seen as true also for nonlinear analysis. For an isolated system in linear analysis, the numerical dissipation d can be calculated as

$$d = \frac{1}{2}(\boldsymbol{v}_n^T\boldsymbol{M}\boldsymbol{v}_n + \boldsymbol{u}_n^T\boldsymbol{K}\boldsymbol{u}_n - \boldsymbol{v}^T\boldsymbol{M}\boldsymbol{v} - \boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u}) = \Delta t^2\left(\beta - \frac{\gamma}{2}\right)(\boldsymbol{a}_n + \gamma\Delta\boldsymbol{a})^T\boldsymbol{M}\Delta\boldsymbol{a} + \left(\gamma - \frac{1}{2}\right)\Delta\boldsymbol{x}^T\boldsymbol{K}\Delta\boldsymbol{x}$$

which shows that equality in the stability condition preserves energy. For strict inequality the dissipation is positive, and the argument against dissipative Newmark schemes is that a great amount of the dissipation occurs for low frequencies and thus deteriorates results. This was put forward by Hans Hilber, Thomas Hughes and Robert Taylor in 1977 [6] in their quest for an improved formulation.

Hilber-Hughes-Taylor (HHT)

The presentation of the HHT time integration scheme in [6] can be generalized to nonlinear implicit by modifying the expression for velocity and geometry in the equations of motion as

$$\boldsymbol{v}_\alpha = -\alpha\boldsymbol{v}_n + (1+\alpha)\boldsymbol{v}$$
$$\boldsymbol{x}_\alpha = -\alpha\boldsymbol{x}_n + (1+\alpha)\boldsymbol{x}.$$

Here $\boldsymbol{v}$ is the velocity from the Newmark scheme, and $\boldsymbol{x} = \boldsymbol{x}_n + \Delta\boldsymbol{x}$ is the geometry at time step $n+1$, and the resulting scheme is known as the HHT or α-method. The method is stable for

$$-1/3 \leq \alpha \leq 0,$$

and from [6] it seems that a value of $\alpha = -0.05$ provides a reasonable amount of numerical dissipation. It is also shown that this new method nicely damps out high frequencies while low frequencies are left unaffected, in contrast to the Newmark method. This method is activated in LS-DYNA by **IMASS=1**, **ALPHA=**$\alpha \in]-1{,}0[$, **GAMMA=**γ and **BETA=**β , i.e., α must be a negative fraction.

Bathe

An attempt to improve Newmark without the outset of introducing numerical dissipation was presented by Klaus-Jürgen Bathe [7], which amounts to using a backward Euler scheme every other step as a correction of the Newmark step. The equations for the latter is summarized as

$$\boldsymbol{a} = \frac{(1+\alpha)}{\Delta t}(\boldsymbol{v} - \boldsymbol{v}_n) - \frac{\alpha}{\Delta t_n}(\boldsymbol{v}_n - \boldsymbol{v}_{n-1})$$
$$\boldsymbol{v} = \frac{(1+\alpha)}{\Delta t}\Delta\boldsymbol{x} - \frac{\alpha}{\Delta t_n}\Delta\boldsymbol{x}_n,$$

where $\Delta\boldsymbol{x}_n$ and Δt_n to denote the geometry and time increment from the previous Newmark step. The parameter α is introduced to provide a family of Bathe schemes, while the specific choice of $\alpha = 1/2$ gives the Bathe scheme as originally presented. This method appears promising in the context of rotating structures since it is a three-point method, and thus has the potential of more appropriately representing curvilinear motion. This method is activated in LS-DYNA by **IMASS=1**, **ALPHA=**$\alpha \in]0{,}1[$, **GAMMA=**γ and **BETA=**β , i.e., α must be a positive fraction. A mathematical analysis of the method is provided in [8].

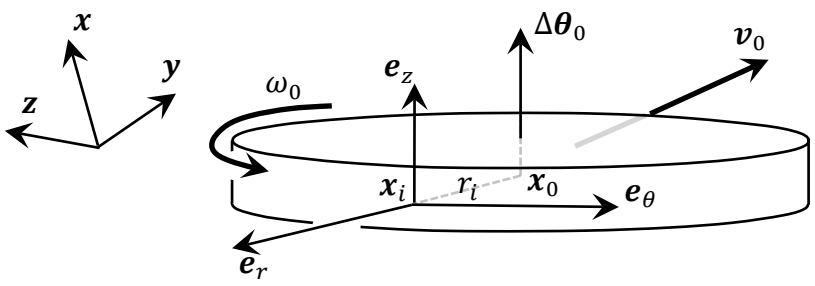

Figure 1 Finite Rotational Dynamics nomenclature, exemplified for a flat rotating disc

Finite Rotational Dynamics (FRD)

The intention with FRD integration is to treat global finite rotations with no loss of accuracy and impose damping on superposed deformations (vibrations) with assumed high frequencies, and this section provides a rough outline of the theory, see Figure 1.

We can estimate the rigid body rotational increment, $\Delta\boldsymbol{\theta}_0$, at the center of mass, $\boldsymbol{x}_0$, by a least square problem that reads[1]

$$\Delta\boldsymbol{x}_i = \Delta\boldsymbol{x}_0 + \Delta\boldsymbol{\theta}_0 \times \boldsymbol{r}_i.$$

Here i ranges over all nodes in the finite element structure, $\boldsymbol{r}_i = \boldsymbol{x}_i - \boldsymbol{x}_0$ is the position vector of node i relative to the center of mass, and $\Delta\boldsymbol{x}_0$ is obtained by direct calculation of the center of masses in current and previous configurations. Without loss of generality, and for the sake of simplifying the exposition, we can assume $\Delta\boldsymbol{\theta}_0^T\boldsymbol{r}_i = 0$. The rotation direction is taken as the axis of a cylindrical coordinate system,

$$\boldsymbol{e}_z = \frac{\Delta\boldsymbol{\theta}_0}{|\Delta\boldsymbol{\theta}_0|},$$

while for each node the other axes are

$$\boldsymbol{e}_r = \frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|}$$
$$\boldsymbol{e}_\theta = \boldsymbol{e}_z \times \boldsymbol{e}_r.$$

Given $\Delta\boldsymbol{x}_0$ and $\Delta\boldsymbol{\theta}_0$, we can use the Newmark scheme to integrate the global translational velocity $\{\boldsymbol{v}_0, \boldsymbol{\omega}_0\}$ and acceleration $\{\boldsymbol{a}_0, \boldsymbol{\alpha}_0\}$ of the global motion of the structure. On top of this, we define[2]

$$\Delta r_i = \boldsymbol{e}_r^T(\Delta\boldsymbol{x}_i - \Delta\boldsymbol{x}_0)$$
$$\Delta\varphi_i = \frac{1}{r_i}\boldsymbol{e}_\theta^T(\Delta\boldsymbol{x}_i - \Delta\boldsymbol{x}_0) - \boldsymbol{e}_z^T\Delta\boldsymbol{\theta}_0$$
$$\Delta z_i = \boldsymbol{e}_z^T(\Delta\boldsymbol{x}_i - \Delta\boldsymbol{x}_0)$$

[1] For finite rotation accuracy, the equation is slightly more involved, but this is irrelevant for understanding the basic concepts of the algorithm.
[2] See previous footnote

which we can use to compute velocities $\{\dot{r}_i, \dot{\varphi}_i, \dot{z}_i\}$ and accelerations $\{\ddot{r}_i, \ddot{\varphi}_i, \ddot{z}_i\}$ with respect to the (curved) cylindrical coordinate system, respectively. We can now transform all this to Cartesian coordinates to obtain

$$\boldsymbol{v} = \boldsymbol{v}_0 + \dot{r}_i \boldsymbol{e}_r + r_i \dot{\theta}_i \boldsymbol{e}_\theta + \dot{z}_i \boldsymbol{e}_z$$
$$\boldsymbol{a} = \boldsymbol{a}_0 + \left(\ddot{r}_i - r_i \dot{\theta}_i^2\right)\boldsymbol{e}_r + \left(2\dot{r}_i \dot{\theta}_i + r_i \ddot{\theta}_i\right)\boldsymbol{e}_\theta + \ddot{z}_i \boldsymbol{e}_z,$$

where $\dot{\theta}_i = \dot{\varphi}_i + \boldsymbol{e}_z^T \boldsymbol{\omega}_0$ and $\ddot{\theta}_i = \ddot{\varphi}_i + \boldsymbol{e}_z^T \boldsymbol{\alpha}_0$. The attraction of this approach lies in perfect force treatment of a rigid body rotation of a deformable structure, which neither of the previously presented algorithms are capable of. It also opens for the possibility to impose damping on the radial (r_i) and axial (z_i) motions, respectively, as well as the overlayed angular motion (φ_i), which we assume contains the high frequencies. The limitation is of course that it is designed specifically for essentially rotating structures and will presumably not add any merit to more general deformation problems.

This method is activated in LS-DYNA by **`IMASS=1`**, **`GAMMA=`**γ and **`BETA=`**β, while **`ALPHA`** is a negative integer. The negative integer specifies the number of independent rotating units in the model, and for each unit a part set is specified that collects all the elements that makes up this unit. As intimated above, using numerical damping to selectively damp higher frequencies leaving low frequencies relatively untouched can be implemented by choosing appropriate values of γ and β. The central idea though is that this will apply to all modes except for the global rotation $\{\boldsymbol{\omega}_0, \boldsymbol{\alpha}_0\}$ which will use the non-dissipative scheme.

Examples

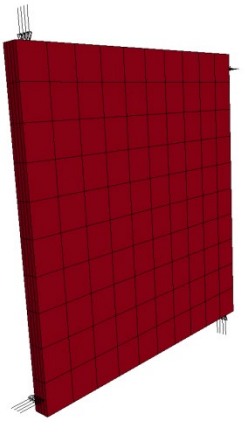

Figure 2 Block subjected to a torque by way of segment load pairs

Rotating Disc

The first example considered was presented in [2], where a torque given by

$$M(t) = \begin{cases} +72\ Nm & 0\ s \le t \le 0.05\ s \\ -72\ Nm & 0.05\ s < t \le 0.1\ s \end{cases}$$

is applied to a square disc with dimensions $V = 0.2 \times 0.2 \times 0.01\ m^3$ and density $\rho = 4429\ kg/m^3$. The disc should spin with constant rotational acceleration to attain a peak rotational velocity of about $304.8\ rad/s$ at $t = 0.05\ s$, and then reverse the motion to end up at rest by $t = 0.1\ s$. The behaviors of the four different methods are illustrated in Figure 3 from which some general conclusions can be drawn. The only two methods able to converge using a time step of $5\ ms$ (corresponding to an angular increment of almost 90 degrees at peak rotational velocity) are FRD and Bathe (using $\alpha = 0.5$, i.e., the original Bathe method), of which the latter suffers from significant numerical dissipation. While FRD also suffers from reduced accuracy it is not as prominent, which is expected by way of algorithm design. As far as the other two, HHT (using $\alpha = -0.05$, i.e., the recommended value) and Newmark (using the trapezoidal rule, energy conserving), both fail using a time step of $2.5\ ms$ while Newmark is also incapable of sustaining a time step of $1\ ms$. Presumably neither of these methods are designed for this degree of nonlinearity, but HHT performs somewhat better due to the slight numerical dissipation. Any of the methods would probably be more stable with more numerical dissipation, which is always possible, but obviously the results would become worse.

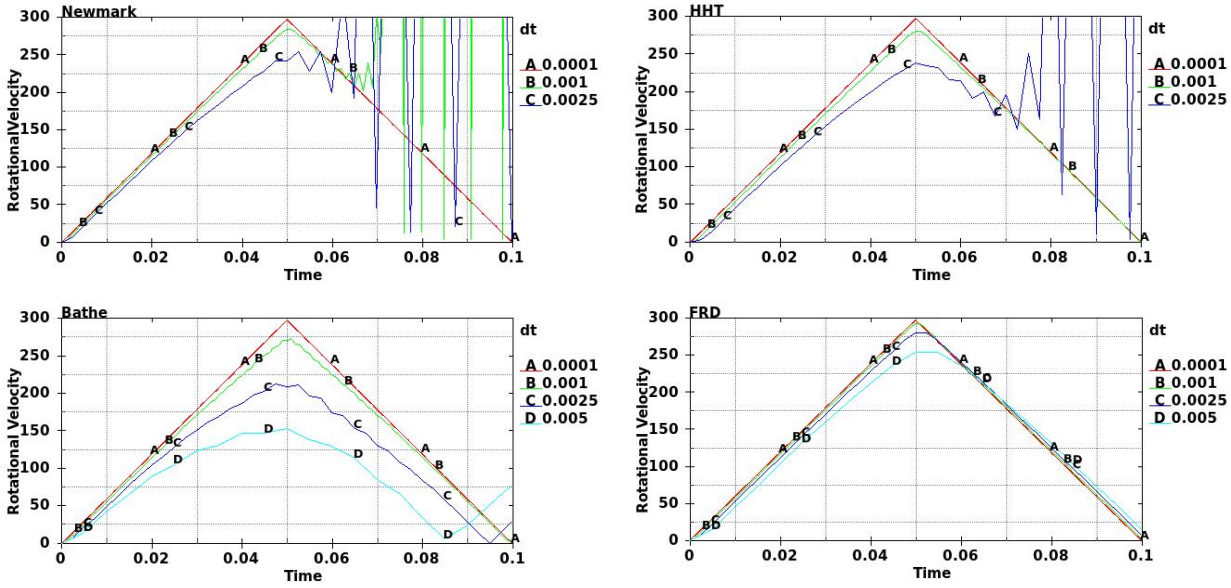

Figure 3 Rotational velocity for acceleration and deceleration of block

As a complement, the torque was changed to

$$M(t) = \begin{cases} +72\ Nm & 0\ s \le t \le 0.05\ s \\ 0\ Nm & 0.05\ s < t \le 0.15\ s \end{cases}$$

with results as shown in Figure 4. Similar conclusions can be drawn, with the additional remark that the zig-zag appearance of the Bathe curves comes from the fact that the scheme toggles between Newmark and Euler scheme each step.

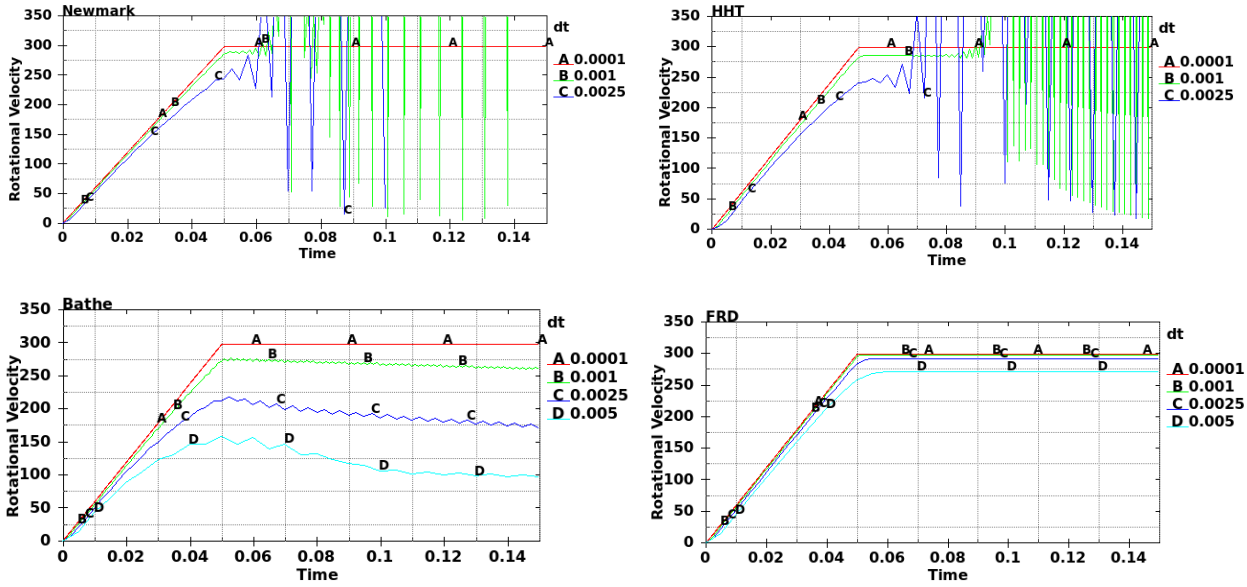

Figure 4 Rotational velocity for acceleration and maintaining velocity of block

Figure 5 Rotating engine prior to blade out

Rotating Engine

A somewhat more interesting example is the rotating engine shown in Figure 5. We impose an initial rotational velocity of 480 rad/s, corresponding to approximately 180000 rpm, on the rotating unit, while the bearing and cover are fixated. The structure is preloaded with the centripetal forces corresponding to this initial condition, and no external loads are acting on the system. By neglecting energy loss in form of damping and friction, the engine should keep rotating with constant velocity indefinitely, and this test shows how the different schemes perform in such a scenario.

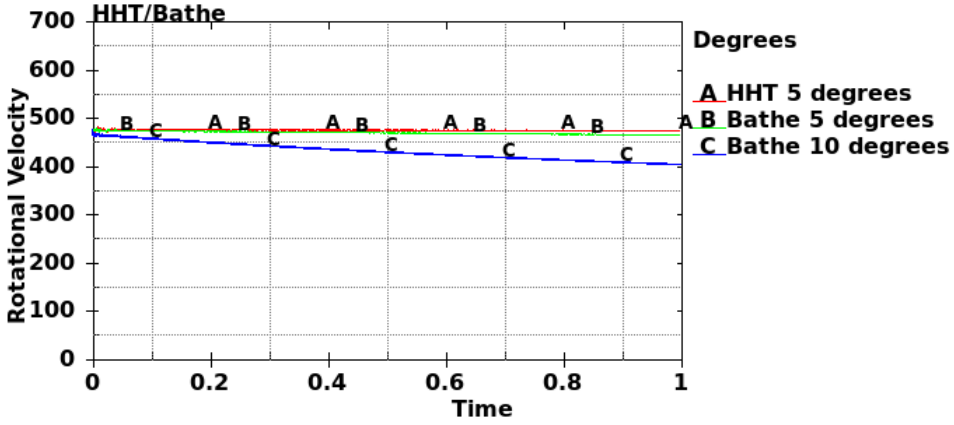

Figure 6 Rotational velocity of engine for HHT and Bathe, using different angle increments

For Newmark and HHT ($\alpha = -0.05$) we made two simulations, each for the duration of $1\ s$. One is using a time step corresponding to an angular increment of 5 degrees and another is targeting 10 degrees. Newmark was unsuccessful in both cases, error terminating quite early. HHT managed to complete the simulation for the first of these simulations, the 5 degree increment, while using 10 degrees also resulted in an early error termination. From curve A in Figure 6 it is evident that HHT damps in a frequency range where the global velocity is not affected significantly, but this slight damping is not enough to render a robust procedure for much larger angular increments. We didn't explore the possibility to add more damping, but maybe this would be a fairly successful approach.

We made the same simulations for the Bathe ($\alpha = 0.5$) scheme, both terminating successfully. However, as seen from curves B and C in Figure 6 the damping is significant and the user needs to assess whether this is a tractable approach. The conclusion here is very much like the one for the rotating disc, the Bathe scheme seems to add more damping to the system than HHT for instance.

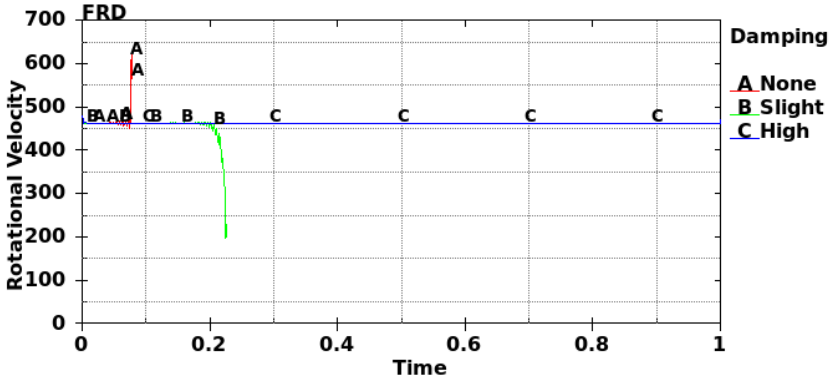

Figure 7 Rotational velocity of engine for Finite Rotational Dynamics, using no, slight and high damping

For FRD we took a different approach, namely to push the angular increment towards 45 degrees in order to demonstrate its ability to represent the dynamics of rotating structures perfectly. To this end, three different simulations were performed, each with different amounts of numerical damping. We label these *None* ($\gamma = 0.5, \beta = 0.25$), *Slight* ($\gamma = 0.55,\ \beta = 0.27563$) and *High* ($\gamma = 0.6,\ \beta = 0.38$) damping, and the results are shown in Figure 7.

It seems that the numerical damping, as intended, provides a stability that allows the highly damped simulation to complete with no reduction in rotational velocity. Important to note is that the angular increment is much higher than for the other simulations, which may explain why the less damped attempts failed. Another observation is that the velocity decreases for the first few time steps, which is correlating with the time step being incremented to reach the target angular increment. This reduction of velocity is not seen when the time step is kept constant and may be something that needs to be looked into.

Summary and Discussion

A brief study of the various implicit time integration schemes offered by LS-DYNA has been conducted, in particular when being applied to fast rotating systems. The aim is to efficiently solve these kinds of problems for long duration with sufficient accuracy, and to this end the following issues arise.

1. What scheme should I use?
2. What parameter settings should I use?
3. What time step should I use?
4. What convergence tolerances should I use?

This paper does not provide a generic and definite answer to any of these, but maybe some ideas for users to start their own investigations. It has been concluded that Newmark does not seem an adequate choice, while HHT and Bathe works reasonably well but adds numerical dissipation that may or may not be acceptable. A new scheme, called FRD, is presented that is capable of preserving rotational speed while adding dissipation to superposed vibrations. We did not investigate the method enough to draw solid conclusions, but it feels promising and we intend to explore its potential.

In this paper, we have been focusing on the quality of the schemes themselves, without too much concern about items 3 and 4 above. For this reason, all numerical examples have been solved with tight tolerances to do away with instabilities caused by numerical errors. Nonetheless, we realize that this is an important practical aspect and hope to delve into this as a continuation. We believe however that high accuracy schemes are less sensitive to numerical errors and can thus sustain larger angular increments and still maintain reasonable quality of results. Using large angular increments will however result in more implicit iterations per time step, so this needs also to be weighed in when targeting the time step size. A suggested starting point would be to aim at an angular increment of 5 to 10 degrees, and use default tolerance settings, and take it from there. Meanwhile we hope to improve the methods of interest by bettering the quality of algorithms and tangent stiffness matrices.

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