Mullins Effect in Rubber
Part Two: Biaxial

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Abstract

The formulation, testing and numerical study of the Mullins effect on rubber are presented. To demonstrate the Mullins effect experimentally, a biaxial test, inflation of a plane circular membrane, is used. Some experimental test data are shown. An approximate solution, a relation between the inflation pressure and the displacement at the center for the inflation of a plane circular membrane is presented. The test data and the approximate solution are used to determine the Mullins damage material constants. These constitution equations are implanted in LS-DYNA. The numerical results from LS-DYNA and analytical results are shown. They agree with one another.

Keywords

Mullins effect, Rubber, Strain-energy-density function, Damage, Biaxial test, LS-DYNA.

Introduction

Rubber mechanics has advanced greatly during the past sixty years. Only a few analytical solutions were obtained before 1960. Later Yang and Feng [1] used IBM360 computers to solve a set of nonlinear differential equations for the inflation of a plane circular membrane. Feng and Huang [2] and Feng and Tielking [3] used UNIVAC 1108 to solve the inflation and contact of rectangular membranes. These were a few examples of using digital computers to solve finite deformation elasticity problems in earlier times.

The constitutive equations for rubber mechanics have also advanced greatly during the past sixty years. Sixty years ago, we had the Mooney [4] constitutive equation for incompressible materials only. Later Ogden [5] extended the Mooney constitutive equation to more terms for incompressible, then to compressible materials. Feng and Hallquist [6] further extended the Ogden formulation to viscoelasticity. Feng [7] developed a large deformation failure criterion for anisotropic materials. A failure criterion for isotropic rubber-like materials has been obtained by Feng and Hallquist [8] as a special case. Feng and Hallquist [9] also obtained another new constitutive equation for the aging of elastomers.

Another new frontier, and an interesting one, of developing a constitutive equation for rubber is the Mullins effect where loading, unloading and subsequent reloading follow different paths. There is very little relaxation or creep; therefore, it is not a viscoelastic phenomenon. It is due to damage of the long molecular chains during loading. Ogden and Roxburgh [10] first modelled the Mullins effect to study unloading in filled rubber. It has been extended by Feng and Hallquist [11] to include the Mullins effect on both unloading and subsequent re-loadings. The new constitutive equation simulates real rubber behavior better. In this paper, we have further extended it to biaxial formulation and testing. Based on the biaxial test results and the new constitutive equation for the Mullins effect, the material constants are determined. The test apparatus and the numerical method for determining these constants are presented in detail. These new constitutive equations are implemented in LS-DYNA. For an equi-biaxial stretching of a cube, the results from LS-DYNA and the analytical results are the same.

The constitutive equations obtained by Feng and Hallquist were all implemented in LS-DYNA. They also developed testing methods for studying the new material models and the numerical method for determining these material constants.

More and more rubber and rubber-like materials have been used in engineering parts and our daily life, demanding more and more accurate modelling of these materials. Developments at Livermore Software Technology Corp. (LSTC) have kept up with this demand.

Formulation

The strain-energy density function with Mullins damage function of a rubber is $\tilde{W}(\lambda_i)$, and

$$\tilde{W}(\lambda_i) = \eta W(\lambda_i) \tag{1}$$

where W is the strain-energy density function based on the initial loading, and $\eta = \eta(W)$ is a damage function for the Mullins effect.

$$\frac{\partial \tilde{W}}{\partial \lambda_i} = \eta \frac{\partial W}{\partial \lambda_i} + W \frac{\partial \eta}{\partial \lambda_i} = \left(\eta + W \frac{\partial \eta}{\partial W} \right) \frac{\partial W}{\partial \lambda_i} \tag{2}$$

The following damage function, a Cauchy first-order ordinary-differential equation, is chosen for this report

For initial loading $$W \frac{\partial \eta}{\partial W} + \eta = 1 \tag{3a}$$

For unloading $$W \frac{\partial \eta}{\partial W} + \eta = 1 - \frac{1}{r_1} \tanh\left[\frac{1}{m_1} \left(1 - \frac{W}{W_m} \right) \right] \tag{3b}$$

For subsequent reloading $W\frac{\partial \eta}{\partial W}+\eta=1-\frac{1}{r_2}\tanh\left[\frac{1}{m_2}\left(1-\frac{W}{W_m}\right)\right]$ (3c)

$W_m(\lambda_i)$ is the maximum strain-energy density function before unloading. r_1, r_2, m_1 and m_2 are the material constants for the Mullins effect damage function. With this damage function, the loading and subsequent unloading follow different paths. For a loading with a value of the strain-energy density function greater than $W_m(\lambda_i)$, the process repeats.

For Mooney-Rivlin materials the strain-energy density equation is:

$$W=C_1(I_1-3)+C_2(I_2-3)=C_1[(I_1-3)+\alpha(I_2-3)] \quad (4)$$

where C_1 and C_2 are material constants and $\alpha=C_2/C_1$. The strain invariants I_1 and I_2 are written in terms of the principal stretch ratios λ_1, λ_2 and λ_3

$$\begin{aligned} I_1 &= \lambda_1^2+\lambda_2^2+\lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2 \end{aligned} \quad (5)$$

An incompressibility condition is assumed in the Mooney-Rivlin material constitutive equation so that

$$\lambda_1\lambda_2\lambda_3=1 \quad (6)$$

The Cauchy stresses (force per unit deformed area) are

$$t_1=\frac{1}{\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_1} \quad (7)$$

and similar equations for t_2 and t_3

For biaxial tension or compression in 1 and 2-directions $\lambda_3=1/\lambda_1\lambda_2$.
Hence,

$$\begin{aligned} I_1 &= \lambda_1^2+\lambda_2^2+1/\lambda_1^2\lambda_2^2 \\ I_2 &= \lambda_1^{-2}+\lambda_2^{-2}+\lambda_1^2\lambda_2^2 \end{aligned} \quad (8)$$

Cauchy stress t_1 is related to the biaxial stretch ratio λ_1.

$$t_1=2C_1\left(\lambda_1^2-\frac{1}{\lambda_1^4}\right)\left(1+\alpha\lambda_1^2\right) \quad (9)$$

and

$$t_2 = 2C_1\left(\lambda_2^2 - \frac{1}{\lambda_2^4}\right)\left(1+\alpha\lambda_2^2\right) \tag{10}$$

An equi-biaxial stretching of a cube is shown in Figure 1.

For equi-biaxial stress $\lambda_1 = \lambda_2 = \lambda$ and $t_1 = t_2 = t$

For initial loading $$t = 2C_1\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(1+\alpha\lambda^2\right) \tag{11}$$

For unloading

$$t = 2C_1\left\{1 - \frac{1}{r_1}\tanh\left[\frac{1}{m_1}\left(1-\frac{W}{W_m}\right)\right]\right\}\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(1+\alpha\lambda^2\right) \tag{12}$$

For subsequent reloading

$$t = 2C_1\left\{1 - \frac{1}{r_2}\tanh\left[\frac{1}{m_2}\left(1-\frac{W}{W_m}\right)\right]\right\}\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(1+\alpha\lambda^2\right) \tag{13}$$

The result for equi-biaxial extension obtained from EXCEL calculations is shown in Figure 2. The material constants are:

$C_1 = 50$, $\alpha = 0.1$, $r_1 = 0.8$, $m_1 = 1.0$, $r_2 = 0.5$ and $m_2 = 5$.

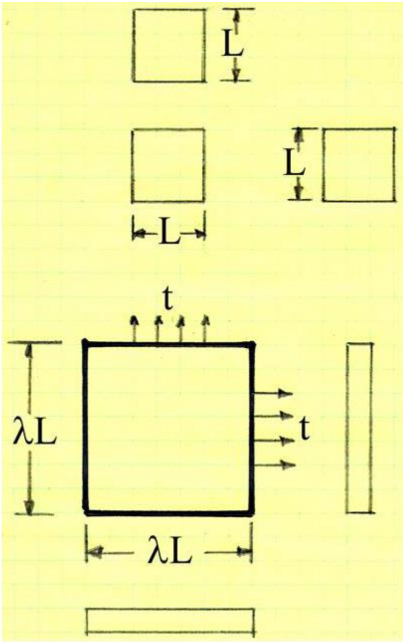

Figure 1. Equi-biaxial stretching of a cube

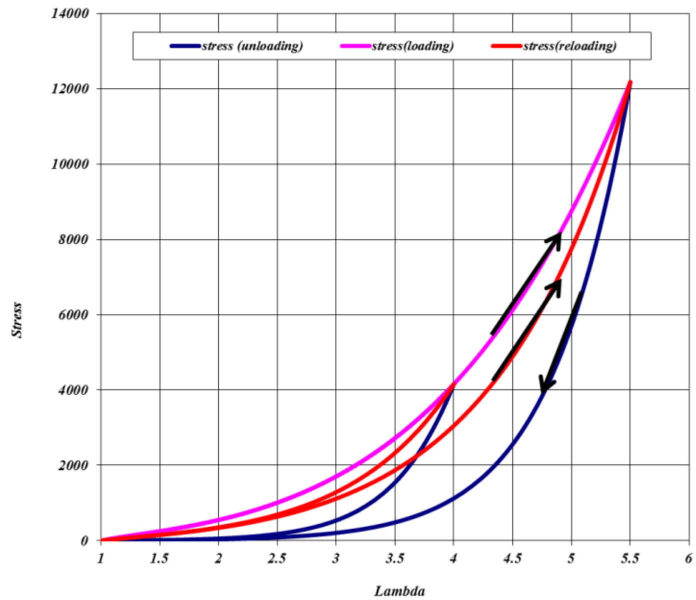

Figure 2. The Mullins effect for equi-biaxial loading, unloading and subsequent reloading, obtained from EXCEL.

The formulation presented in this paper applies to one-, two- and three-dimensional problems. For general three-dimensional problems the mathematical formulation has been implemented in LS-DYNA.

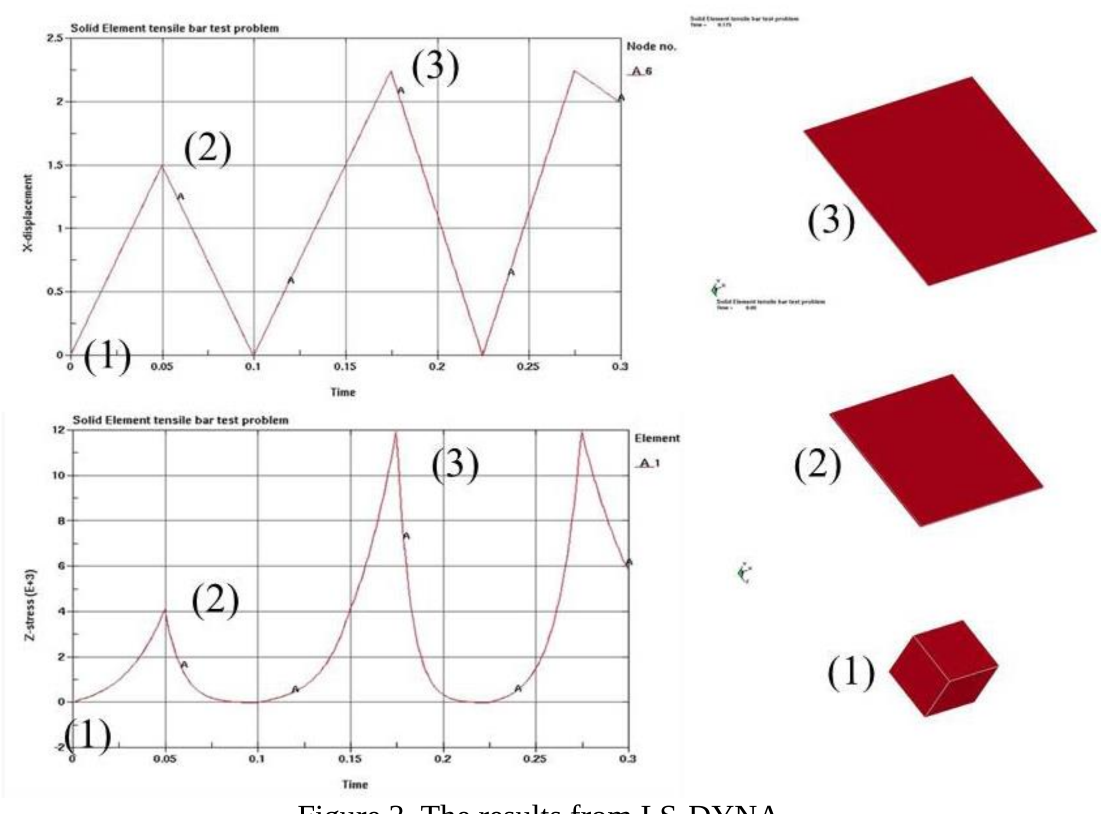

Figure 3. The results from LS-DYNA.

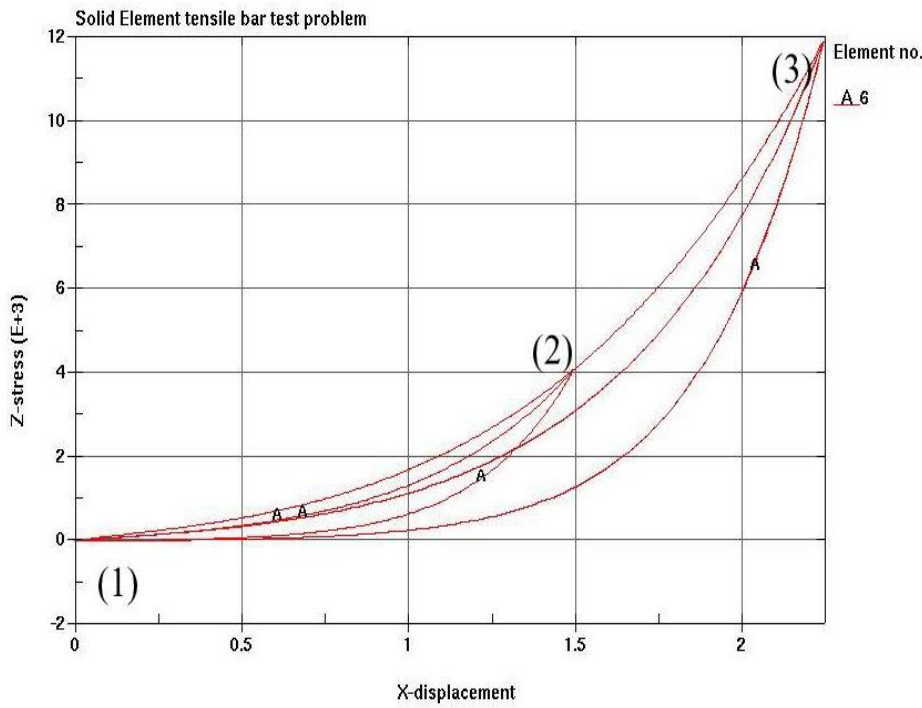

Figure 4. The stress-displacement plot from LS-DYNA.

The result of a cube of 0.5x0.5x0.5 subjected to equi-biaxial extension is obtained from LS-DYNA. The displacement and the stress in the cube are shown in Figure 3. The material constants used in the analytical study, Figure 2, are used in the LS-DYNA calculations. The stress-displacement plot is shown in Figure 4. The displacement can be converted to the stretch ratio λ. The results shown in Figure 2 from the analytical calculation and Figure 4 from LS-DYNA are the same.

Approximate Solution for Inflating a Circular Membrane

The numerical solution for studying the inflation of a thin circular membrane, shown in Figure 5, has been obtained by Yang and Feng [1]. However, using the numerical method for determining the material constants will be cumbersome. Here we used the approximate solution of inflating pressure and deformation at the pole, by Christensen and Feng [12], to determine the material constants C_1 and α. The result of the approximate solution is outlined here. The approximate relationship between the inflating pressure (P) and the deformation at the pole (Δ) is

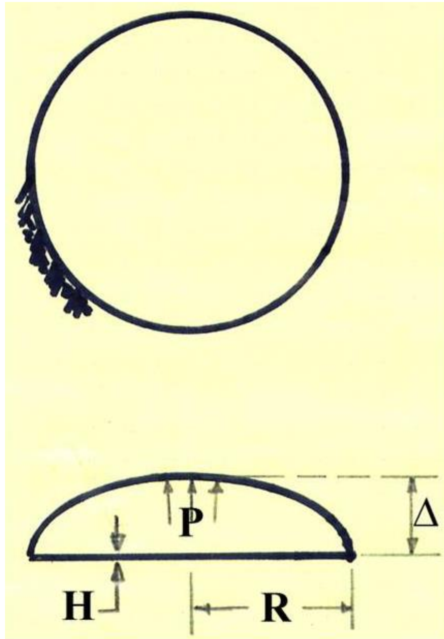

Figure 5. Inflation of a plane circular membrane

$$P = \frac{4C_1H}{R}\left\{\frac{2}{\left(\frac{\Delta}{R}+\frac{R}{\Delta}\right)}\left[\left(1-\frac{1}{\lambda^6}\right)+\alpha\left(\lambda^2-\frac{1}{\lambda^4}\right)\right]\right\} \tag{14}$$

The relationship between λ and Δ is

$$\lambda = \frac{\left(\frac{\Delta}{R}+\frac{R}{\Delta}\right)}{2}\sin^{-1}\left(\frac{2}{\left(\frac{\Delta}{R}+\frac{R}{\Delta}\right)}\right) \tag{15}$$

where R is the initial radius and H is the initial thickness of the circular membrane.

These relationships can be extended to the inflation of a plane circular membrane with Mullins effect.

For initial loading

$$P = \frac{4C_1H}{R}\left\{\frac{2}{\left(\frac{\Delta}{R}+\frac{R}{\Delta}\right)}\left[\left(1-\frac{1}{\lambda^6}\right)+\alpha\left(\lambda^2-\frac{1}{\lambda^4}\right)\right]\right\} \tag{16}$$

For unloading

$$P = \frac{4C_1H}{R}\left\{\frac{2}{\left(\frac{\Delta}{R}+\frac{R}{\Delta}\right)}\left[\left(1-\frac{1}{\lambda^6}\right)+\alpha\left(\lambda^2-\frac{1}{\lambda^4}\right)\right]\right\}*\left\{1-\frac{1}{r_1}\tanh\left[\frac{1}{m_1}\left(1-\frac{W}{W_m}\right)\right]\right\} \quad (17)$$

For subsequent reloading

$$P = \frac{4C_1H}{R}\left\{\frac{2}{\left(\frac{\Delta}{R}+\frac{R}{\Delta}\right)}\left[\left(1-\frac{1}{\lambda^6}\right)+\alpha\left(\lambda^2-\frac{1}{\lambda^4}\right)\right]\right\}*\left\{1-\frac{1}{r_2}\tanh\left[\frac{1}{m_2}\left(1-\frac{W}{W_m}\right)\right]\right\} \quad (18)$$

Biaxial Test

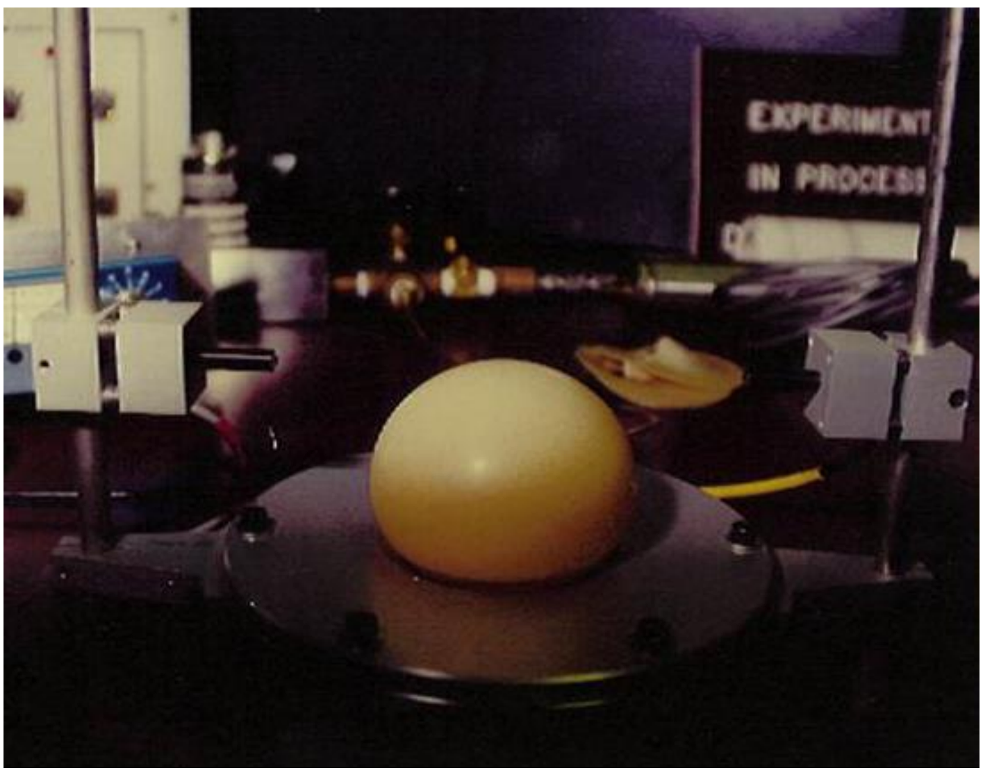

Figure 6. The apparatus

In the experiment a flat circular membrane of rubber was clamped between two plates as shown in Figure 6. The membrane is inflated by air or liquid from a reservoir at a constant temperature. The height of the deformed membrane at the pole is measured by a LVDT. The pressure is measured with a pressure transducer. The pressure-height relationship is measured, as shown in Figure 7. The radius of the membrane is 2 inches and the thickness is 0.015 inches. In the experiment, the membrane is inflated monotonically to 1.6 inches then deflated to the flat surface. It is inflated again to 1.6 inches then deflated again for three cycles. During the loading and unloading, the deformation, at the pole, is in a uniform biaxial stress state.

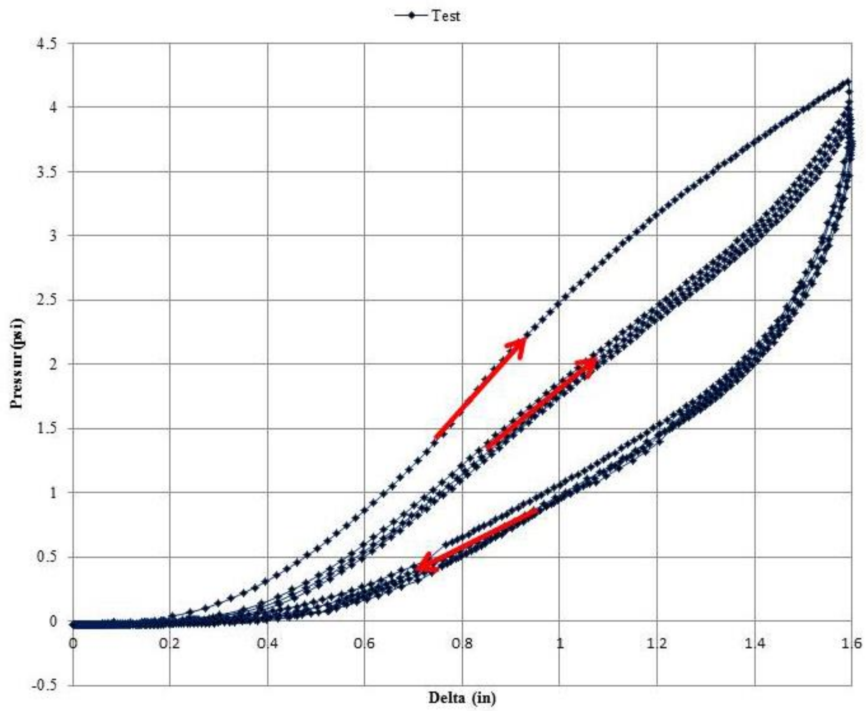

Figure 7. The test data for the pressure-height relationship

Determining the damage constants

The material constants can be obtained from test data and the least-square error minimization method. At i^{th} datum, the measured displacement at the pole is $\Delta(i)$, the measured pressure is $\tilde{P}[\Delta(i)]$, and the calculated pressure based on Eqs. (16 -18) is $P[\Delta(i)]$. Hence, the error between measured pressure ($\tilde{P}$) and the calculated pressure (P) at i^{th} data point is

$$P[\Delta(i)] - \tilde{P}[\Delta(i)] \tag{19}$$

For m data points, the sum of the square of errors is

$$S = \sum_{i=1}^{m} \left\{ P[\Delta(i)] - \tilde{P}[\Delta(i)] \right\}^2 \tag{20}$$

By minimizing the sum of the squares of errors S , the material constants C_1, α , r_1 , m_1 , r_2 and m_2 are determined. They are three-decoupled minimization sets: C_1, and α can be determined from the initial loading curve; r_1 and m_1 can be determined from the unloading curve; r_2 and m_2 can be determined from the reloading curve. The determined Mooney-Rivlin constants are: $C_1 = 5$ and $\alpha = 0.01$. The determined Mullins damage constants are: $r_1 = 1.65$, $m_1 = 0.35$, $r_2 = 3.9$ and $m_2 = 0.4$. The best-fit initial loading, unloading and reloading pressure-deformation curves are shown in Figure 8.

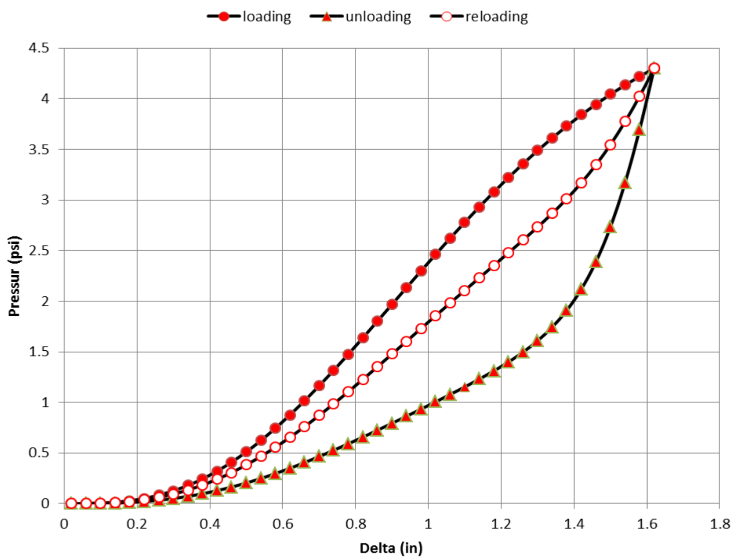

Figure 8. The best-fit initial loading, unloading and reloading pressure-deformation curves

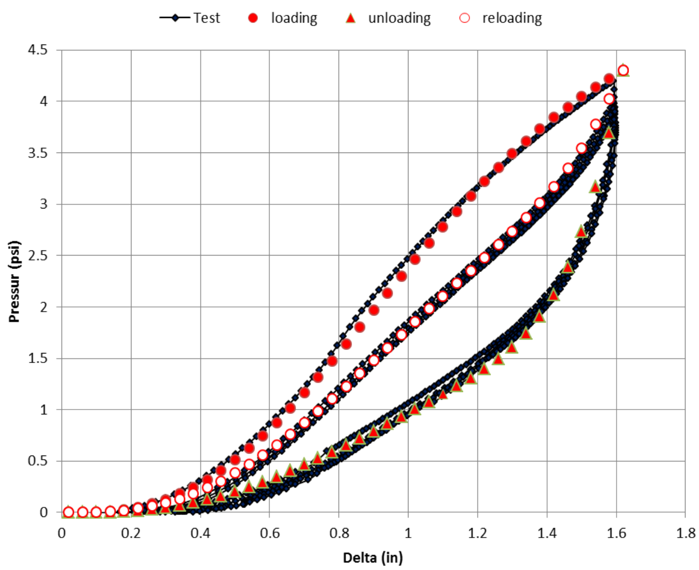

Figure 9. The best-fit initial loading, unloading and reloading pressure-deformation curves and the test data

The best-fit curves that show the Mullins effect and the test data are shown in Figure 9. They agree with one another. It also shows that the theory, the test and the numerical formulation work well.

Future work

The formulations and applications can be extended to various rubbers with strain-energy density represented by various constitutive equations such as: neo-Hookean, Mooney, Ogden incompressible, and Ogden compressible materials. It can also been extended to viscoelastic materials for compressible and incompressible viscoelastic materials subjected to very large deformation.

We assumed that the Mullins damage function in this paper is represented by a hyperbolic tangent function; it can be changed to other functions if needed**.**

Acknowledgments

W. W. Feng dedicates this paper to his mother Mrs. Ke Ying Feng (馮葛櫻女士).

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